The Spectral Method for the Synthesis of Integral H-sequences with an Ideal Periodic Autocorrelation Function

Andrey Sadchenko and Oleg Kushnirenko

Department of Radio electronic and Telecommunication Systems, Institute of Information Security, Electronics and Telecommunications, Odessa National Polytechnic University, Shevchenko Avenue, 1, Odessa, Ukraine 65044.

koa@opu.ua

***Abstract*— In problems of signal discrimination, orthogonal cyclic sequences with good periodic autocorrelation functions (PACF) are most widely used. There are published studies on binary cyclic sequences with low PACF side lobes that do not possess the property of orthogonality for a length of N > 4. Hence, we propose the systems of orthogonal multilevel integer cyclic sequences of N = mn length, for which the cyclic convolution coincides with the m-convolution (H-sequence), which can be very effectively calculated. It is proposed to use amplitude phase shift keying (APSK) to transmit the H-sequence over the communication channel. The purpose of this study is to develop a regular algorithm for the synthesis of H-sequences of arbitrary length N = mn based on an analysis of their spectral properties. Studying the properties of the twiddle factors of the Fourier transform matrix in the basis of the Vilenkin-Chrestenson function allowed for developing a synthesis method in the frequency domain of multilevel integer H-sequences with ideal periodic autocorrelation functions.**

***Index Terms*—Amplitude-Phase Shift Keyed (APSK) Signal; Binary Phase Shift Keying; Cyclic Convolution; Integer H-Sequence.**

I. INTRODUCTION

The intensive development of discrete messaging systems has led to the development of new combined modulation techniques. Along with quadrature amplitude modulation (QAM), a great interest has recently been shown in amplitude-phase-shift keyed (APSK) signals, which allow the usage of more efficient synchronization schemes, since in this case, there are no problems with the phase ambiguity of the received signal [1], [2].

APSK signals can be demodulated using a two-stage algorithm, the first stage of which is used to isolate the video frequency sequence, and the second is used to make a decision due to decoding "as a whole". In many cases, however, it is rather difficult to implement the optimal reception of APSK signals reception "as a whole" [3], [4]. The optimal "as a whole" decoder scheme can be significantly simplified by using fast *m*-convolution algorithms in the basis of the Vilenkin-Chrestenson function [5], [6] or in particular cases, dyadic convolution [7-10].

In order for the circular convolution vector of a discrete signal encoded by a numerical sequence $H = \{h(i)\}$, $i = \overline{0, N-1}$, $N = m^n$ (where *m* is the *m*-shift base value, *h(i)* is a complex number in the general case) to coincide with the *m*-convolution vector, it is necessary and sufficient that the elements of coding H-sequence satisfy the following system of limitation (1). For each $i = \overline{1, m-1}$, the elements are:

$$\begin{cases} h(i) = h(i+lm), \quad l = \overline{1, m^{n-1}-1}; \\ h(im) = h(im+lm^2), l = \overline{1, m^{n-2}-1}; \\ h(im^2) = h(im^2+lm^3), l = \overline{1, m^{n-3}-1}; \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ h(im^{n-2}) = h(im^{n-2}+lm^{n-1}), l = \overline{1, m-1}; \\ \text{for each } l = \overline{1, m^{n-1}-1}, \text{ elements } h(im^{n-1}) - \text{not related to restrictions} \end{cases} \quad (1)$$

The authors of [5] have shown that the circular convolution vector in problems of distinguishing a normal system of orthogonal cyclic signals invariant to *m*-shift can be very effectively calculated using the fast direct method of calculating convolution in the time domain, based on the structural properties of the processed signals.

Note, however, that at present time, the frequency properties of discrete APSK signals encoded by H-sequences are not fully studied. Specifically,

- regular H-sequence synthesis algorithms with integer elements for an arbitrary value of the shift m-base have not yet been developed;
- there are no recommendations on the implementation of information transmission systems based on orthogonal cyclic APSK signals encoded by H-sequences.

The purpose of this work is to study the time-frequency properties and to develop a regular method for the synthesis in the frequency domain of discrete APSK signals based on integer H-sequences with an ideal periodic autocorrelation function.

II. H-SEQUENCE PROPERTIES IN THE TEMPORARY AREA

Numeric sequences $H = \{h(i)\}$, $i = \overline{0, N-1}$, $N = m^2$ that are invariant to m-shift and constructed in accordance with rule (1) will be called H-sequences, and the number Ψ of the various elements, which makes up the H-sequence -- degrees of freedom.

By definition (1), the element *h(NmodN) = h(0)* in the H-sequence, and therefore the choice of its value is not restricted. Similarly, the values of $h(im^k)$, where $i = \overline{1, m-1}$ for each $k = \overline{0, n-1}$, are also selected independently of each other. We shall call these elements "generators", and all other elements of the H-sequence will be called "redundant", because their values exactly repeat the values of the corresponding generator elements as indicated in (1). The H-sequence will be called q-ary, if the base of its generating elements is q.

The definitions above allow establishing some of the properties of the H-sequence and its elements:

1. The number of generators (2) and redundant (3) elements of the H-sequence:

$$\psi_m = n(m-1)+1 \quad (2)$$

$$r = N - \psi_m = m^n - n(m-1) - 1 \quad (3)$$

2. The periodicity property of the forming elements. Each generator element of the form $\{h(im^k)\}$, where $i = 1, m-1$ for each $k = \overline{0, n-2}$ occurs in the H-sequence exactly m^{n-k-1} times with a period $T_k = m^{k+1}$, and each generator element of the form $h(im^{n-1})$, $i = \overline{0, m-1}$ occurs only once.
3. The H-sequence can have Ψ degrees of freedom, where the value is:

$$1 \le \Psi \le \Psi_m \quad (4)$$

4. The number of q-ary H-sequences is determined by the relation:

$$V = \sum_{\psi=1}^{\psi_m} q^{\psi} \quad (5)$$

For example, for $N = m^n = 2^3 = 8$, the general structure of the H-sequences has the form $H = \{h(0), h(1), h(2), h(3), h(4), h(5), h(6), h(7)\}$, where $h(1) = h(3) = h(5) = h(7)$, $h(2) = h(6)$ and the elements $h(0)$ and $h(4)$ are not restricted; parameter $1 \le \Psi \le 4$; the number of ternary (q=3) H-sequences V = 120, while the total number of all possible ternary sequences $V_0 = 6551$.

Each N-long sequence from set (5) makes it possible to construct (encode) a normal system of $L = N$ cyclic signals that allow fast circular convolution according to system (1) using the direct method [5], which increases the efficiency of the decoding device performing reception «as a whole».

From both theoretical and practical standpoints, however, the most interesting are the H-sequences that have predetermined correlation properties, and particularly those with ideal periodic autocorrelation functions (PACF) with the sidelobe values of:

$$R(\tau) = \sum_{i=0}^{N-1} h(i)h((i-\tau) \bmod N) = 0 \text{, for } \tau = \overline{1, N-1} \quad (6)$$

It is obvious that H-sequences with ideal PACFs allow constructing normal systems of q-ary orthogonal cyclic signals invariant to the m-shift.

Consider the symmetry property of the H-sequence. An arbitrary N-long numerical sequence H is called symmetric if its elements satisfy the following condition:

$$h(i) = h(N-i), \quad i = \overline{0, N-1} \quad (7)$$

Taking property 2 into account, in order to establish the symmetry of an H-sequence, it is sufficient to accurately verify the fulfillment of condition (7) only for its generating elements of the form:

$$h(im^k) = h[N - m^k(m-i)], \quad i = \overline{1, m-1}, \quad k = \overline{0, n-1} \quad (8)$$

Since taking into account the reduction of the arguments modulo N, the element $h(0) = h(N-0)$. Analysis of relation (8) allows to establish the following properties of the symmetry of the H-sequences.

5. H-sequences with the magnitude of the base of the shift $m = 2$ are always symmetric. Indeed, for $m = 2$, from (8) immediately obtain:

$$h(2^k) = h(N - 2^k), k = \overline{0, n-1} \quad (9)$$

6. H-sequences with the value of shift base $m > 2$ may be symmetrical only in those cases, where the number of degrees of freedom ψ less than the number of generator elements ψ_m and lies in the range:

$$1 \le \psi \le \psi_c, \text{ where } \psi_c = \begin{cases} nm/2+1, & m - even \\ n(m-1)/2+1, & m - odd \end{cases} \quad (10)$$

Proof: Let the H-sequence with a base of shift $m > 2$ be symmetric, then it follows from (8) that $h[N - m^k(m-i)] = h[m^k(m-i)]$. In other words, the equality of the constituent elements must be fulfilled: $h(im^k) = h[(m-i)m^k]$, $i = \overline{0, m-1}$, $k = \overline{0, n-1}$, which leads to decreasing the number of degrees of freedom $\psi < \psi_m$. If m is an even number, then there always exists such i that $i = m-i$, and, therefore, for a given $i = m-i$, the number of degrees of freedom does not decrease. When m is odd, equality $i = m-i$ is impossible, which completes the proof of property 6.

III. Properties of H-Sequences in the Frequency Area

Discrete Fourier transform (DFT) of an arbitrary numerical sequence $h = \{h(i)\}$, $i = \overline{0, n-1}$ is determined by the expression (11):

$$H(k) = \sum_{i=0}^{N-1} h(i) W^{ik}, \quad k = \overline{0, N-1} \quad (11)$$

where: $W = e^{-j2\pi/N}$ = Twiddle factor

The set of coefficients $H(k)$, $k = \overline{0, N-1}$ will be called the DFT spectrum of the sequence $\{h(i)\}$. The general properties of the DFT spectrum are well studied, for example [11]. Let us specify a number of properties of the DFT spectrum for the H-sequences considered in this paper.

1. DFT spectrum of a real symmetric sequence $H = \{h(i)\}$, $i = \overline{0, N-2}$ is also a real symmetric sequence.

$$H(k) = H(N-k), k = \overline{0, N-1} \quad (12)$$

Proof: For an arbitrary real sequence where in the derivation, the properties of the complex conjugation "*" operation and the equality $h^*(i) = h(i)$ and $(W^{ik})^* = W^{-ik}$ are taken into account:

$$H(k) = \sum_{i=0}^{N-1} h(i)W^{ik} = \left[\left[\sum_{i=0}^{N-1} h(i)W^{ik}\right]^*\right]^* = H^*(N-k) \qquad (13)$$

DFT spectrum of an arbitrary symmetric sequence where the output takes into account the symmetry property of the elements: $h(i) = h(N-i)$, $i = \overline{0, N-1}$ and property of twiddle factors: $W^e = W^{-(N-e)}$ for arbitrary integers e:

$$H(k) = \begin{cases} h(0) + h(N/2) + 2\sum_{i=0}^{N/2-1} h(i)\cos(2\pi ik/N), \ N-even \\ h(0) + 2\sum_{i=1}^{(N-1)/2} h(i)\cos(2\pi ik/N), \ N-odd \end{cases} \qquad (14)$$

Expression (14) makes it clear that if the sequence $\{h(i)\}$ is real, then its DFT-spectrum is also real. Combining the results of (13) and (14), we establish the validity of (12).

Let us now consider the properties of the DFT spectrum of the H-sequences, i.e., numerical sequences invariant to the m-shift. In order to do this, let us take into account the decomposability property of the number $N = m^n$ and properties 2 (section II) of the periodicity of the constituent elements of the H-sequence and present the expression for the DFT coefficients (11) in the form of n terms -- partial sums.

$$H(k) = \sum_{i=0}^{N-1} h(i)W^{ik} = \sum_{e=1}^{n-1} Ie(k) + In(k), k = \overline{0, m^n - 1} \qquad (15)$$

Where the partial sums:

$$Ie(k) = \sum_{i_e=1}^{m-1} h(i_e m^{e-1}) = \sum_{i_e=0}^{m^{n-e}} W^{(i_e m^{e-1} + I_e m^e)k}, e = \overline{1, n-1} \qquad (16)$$

$$In(k) = \sum_{In=0}^{m-1} h(i_n m^{n-1}) W^{i_n m^{n-1} k} \qquad (17)$$

Denote the internal sum in (16) as $Se(k)$ and represent it in the form:

$$Se(k) = W^{i_e m^{e-1} k} \sum_{i_e=0}^{m^{n-e}-1} W^{I_e m^e k}, e = \overline{1, n-1} \qquad (18)$$

The set of vectors under the sum sign in (18) with $k = 1$ will be called the initial system of vectors. Then, the sequence of exponents of the initial system of m^{n-e} twiddle vectors for each $i_e = \overline{0, m-1}$ is represented as:

$$\{I_e m^e\}, I_e = \overline{0, m^{n-e} - 1} \qquad (19)$$

The rule for converting exponents (19) for each $k = \overline{0, N-1}$ has the form:

$$\{I_e m^e k\}, I_e = \overline{0, m^{n-e} - 1} \qquad (20)$$

A comparison of (19) and (20) shows that the transformation mechanism resembles decomposition of the elements of a group by subgroups into adjacent classes. The sum (18) of the initial vector system for each $e = \overline{1, n-1}$ and each $i_e = \overline{0, m-1}$ equals to zero as the resultant of m^{n-e} twiddle vectors symmetrically located on the unit circle. The transformation rule (20) converts the original system of twiddle vectors with exponents from (19) into another -- with exponents from (20). Considering the property of twiddle vectors periodicity $w^b = w^{b \bmod N}$, $N = m^n$, let us find the specific values of the function $Se(k)$ for different e and k:

$$Se(k) = \begin{cases} m^{n-e}, if \ (k,m) > 1 \ and \ m^{n-e+1} \setminus k; \\ m^{n-e} W^{i_e m^{e-1} k}, if \ (k,m) > 1 \ and \ m^{n-e} \setminus k; \\ 0, \ if \ (k,m) = 1 \ or \ (k,m) > 1 \ and \ m^{n-e} \times k. \end{cases} \qquad (21)$$

In (21), the following notation is used: (a,b) – the greatest common divisor of two integers a and b; $b \setminus a$ – b is the divisor of a; $b \times a$ – b is not the divisor of a.

Note that the values of the function $Se(k)$ do not depend on the specific type of the H-sequence; therefore, it is advisable to tabulate this function and then reuse the tabulation results to calculate the DFT spectrum of arbitrary H-sequences of a given length. For example, for $N = m^n = 3^3 = 27$ all values of the function $Se(k)$, $e = \overline{1, 2}$, $k = \overline{0, 26}$ are presented in Table 1, which is essentially a sparse matrix.

Let us demonstrate that the functions $Se(k)$ and $In(k)$ have the property of invariance to the m-shift. To do this, let us uniquely represent the variable $k = \overline{0, N-1}$, $N = m^n$, similar to variable I representation from (1), through the parameters $v = \overline{1, n-1}$, $k = \overline{1, m-1}$, and $I_v = \overline{0, m^{n-v} - 1}$ as:

$$k = k_v m^{v-1} + I_v m^v \text{ or } k = k_n m^{n-1} \qquad (22)$$

where: $k_n = \overline{0, m-1}$

Table 1
Discrete Fourier Transform Factors of H-sequences of Length $N = m^n = 3^3 = 27$

	k								
e	0	1	2	3	4	5	6	7	8
1	9	0	0	0	0	0	0	0	0
2	3	0	0	$3W^{i}{}_{2}{}^{9}$	0	0	$3W^{i}{}_{2}{}^{18}$	0	0
	k								
e	9	10	11	12	13	14	15	16	17
1	$3W^{i}{}_{1}{}^{9}$	0	0	0	0	0	0	0	0
2	3	0	0	$3W^{i}{}_{2}{}^{9}$	0	0	$3W^{i}{}_{2}{}^{18}$	0	0
	k								
e	18	19	20	21	22	23	24	25	26
1	$9W^{i}{}_{1}{}^{18}$	0	0	0	0	0	0	0	0
2	3	0	0	$3W^{i}{}_{2}{}^{9}$	0	0	$3W^{i}{}_{2}{}^{18}$	0	0

Considering the property of the periodicity of the turning factors, we immediately find that:

$$Se(k_v m^{v-1}) = Se(k_v m^{v-1} + I_v m^v), \ \ I = \overline{0, m^{n-v} - 1} \qquad (23)$$

For this value $Se(k_n m^{n-1})$ at $k = \overline{0, m-1}$ in general has different numbers:

$$In(k_v m^{v-1} + I_v m^v), I = \overline{0, m^{n-v} - 1} \qquad (24)$$

The above analysis allows for establishing the validity of the following property of the DFT spectrum.

2. DFT spectrum of H-sequence of length $N = m^n$ invariant to

m-shift is also a sequence invariant to m-shift, i.e. for each $v = \overline{1, n-1}$ and $k_v = \overline{1, m-1}$, DFT coefficients $H(k_v m^{v-1}) = H(k_v m^{v-1} + I_v m^v)$, $I_v = \overline{0, m^{n-v} - 1}$; for each $k_n = \overline{0, m-1}$, DFT coefficients are not related to restrictions.

Indeed, the expression for the DFT coefficients (11), taking into account (16) and (18), are expressed as:

$$H(k) = \sum_{e=1}^{n-1} \sum_{i_e=1}^{m-1} h(i_e m^{e-1}) Se(k) + In(k). \tag{26}$$

Now, taking into account the relations of (23) and (24), we can immediately establish the validity of property 2 (section III).

3. Forming DFT spectrum elements $H(k_n m^{n-1})$, $k_n = \overline{0, m-1}$ and $H(k_v m^{v-1})$, $v = \overline{1, n-1}$, $k_v = \overline{1, m-1}$ of the symmetric H-sequence of length $N = m^n$ are bound by the conditions:

$$\begin{cases} H(k_n m^{n-1}) = H[(m - k_n) m^{n-1}], \\ H(k_v m^{v-1}) = H[(m - k_v) m^{v-1}]. \end{cases} \tag{27}$$

4. The maximum possible number of degrees of freedom for the symmetric DFT spectrum of a symmetric H-sequence of length $N = m^n$ is determined by the relations (28) for $m = 2$ and (29) for $m > 2$:

$$\psi_c = n+1 \tag{28}$$

$$\psi_c = \begin{cases} nm/2 + 1, & m - even, \\ n(m-1)/2 + 1, & m - odd. \end{cases} \tag{29}$$

Properties 3 and 4 (from Section III) follow from properties 5 and 6 (from Section II) of the generating elements of the symmetric H-sequence.

IV. ARITHMETIC COMPLEXITY OF CALCULATION OF THE DFT-SPECTRUM OF H-SEQUENCES

Let us estimate the arithmetic complexity of the calculation of the DFT spectrum of an arbitrary H-sequence. Analysis of (25) shows that, in order to calculate the DFT spectrum, it is necessary to calculate not more than $\psi_m = n(m-1) + 1$ DFT coefficients, which form the whole spectrum. These ψ_m coefficients are: $H(k_n m^{n-1})$, $k_n = \overline{0, m-1}$ and $H(k_v m^{v-1})$, $v = \overline{1, n-1}$, $k_v = \overline{1, m-1}$.

For example, in a case $N = m^n = 3^3 = 27$, it is necessary and sufficient to find the DFT of $H(k)$ for the values of the coefficients $k \in \psi_m$ where the multiplicity ψ_m = (0, 1, 2, 3, 6, 9, 18). From (15), it follows that when $n = 3$, the coefficients are $H(k) = I_1(k) + I_2(k) + I_3(k)$. Taking into account the property (21) of the function $Se(k)$ and the data in Table 1, we can find specific expressions for partial sums $Ie(k)$, $e = \overline{1, n-1}$ and for the partial sum $In(k)$ (26). These expressions are presented respectively in Table 2, 3 and 4.

Note that Table 2–4 show the equality (=) and the inequality of the corresponding partial sums (≠) to zero for the general case of arbitrary H-sequences of length $N = 3^3 = 27$.

Table 2
Partial Sums of Elements of H-sequences of Length $N = 3^3 = 27$

$I_1(0)$	$= h(1)9 + h(2)9$	$\neq 0$
$I_1(1)$	$= h(1)0 + h(2)0$	$= 0$
$I_1(2)$	$= h(1)0 + h(2)0$	$= 0$
$I_1(3)$	$= h(1)0 + h(2)0$	$= 0$
$I_1(6)$	$= h(1)0 + h(2)0$	$= 0$
$I_1(9)$	$= h(1)9W^9 + h(2)9W^{18}$	$\neq 0$
$I_1(18)$	$= h(1)9W^{18} + h(2)9W^9$	$\neq 0$

Table 3
Partial Sums of Elements of H-sequences of Length $N = 3^3 = 27$

$I_2(0)$	$= h(3)3 + h(6)3$	$\neq 0$
$I_2(1)$	$= h(3)0 + h(6)0$	$= 0$
$I_2(2)$	$= h(3)0 + h(6)0$	$= 0$
$I_2(3)$	$= h(3)3W^9 + h(6)3W^{18}$	$\neq 0$
$I_2(6)$	$= h(3)3W^{18} + h(6)3W^9$	$\neq 0$
$I_2(9)$	$= h(3)3 + h(6)3$	$= I_2(0)$
$I_2(18)$	$= h(3)3 + h(6)3$	$= I_2(0)$

Table 4
Partial Sums of Elements of H-sequences of Length $N = 3^3 = 27$

$I_3(0)$	$= h(0) + h(9) + h(18)$	$= 0$
$I_3(1)$	$= h(0) + h(9)W^9 + h(18)W^{18}$	$\neq 0$
$I_3(2)$	$= h(0) + h(9)W^{18} + h(18)W^9$	$\neq 0$
$I_3(3)$	$= h(0) + h(9) + h(18)$	$= I_3(0)$
$I_3(6)$	$= h(0) + h(9) + h(18)$	$= I_3(0)$
$I_3(9)$	$= h(0) + h(9) + h(18)$	$= I_3(0)$
$I_3(18)$	$= h(0) + h(9) + h(18)$	$= I_3(0)$

For arbitrary $N = m^n$ which is the lengths of H-sequences, we obtain n tables similar to Table 2–4, with each table, according to property (21), containing exactly m partial sums not equal to zero, the remaining partial sums either equal to zero, or coincide with the partial sum $Ie(0)$. Therefore, the upper bound for estimating the number of multiplication and addition operations for two generally complex numbers required to calculate the DFT of the H-sequence spectrum of length $N = m^n$ is defined as:

$$\Theta \leq nm(m-1) \tag{30}$$

Which is significantly less than the known estimate $N \log_2 N$, which is obtained without taking into account the structural properties of time sequences.

V. METHOD OF SYNTHESIS OF H-SEQUENCES WITH IDEAL PACF AND INTEGRAL ELEMENTS

Suppose $E(k)$ is the energy DFT spectrum of the H-sequence of length $N = m^n$ with ideal PACF (6) limited, without generality loss, by the value of N. Then, in accordance with the Wiener–Khinchin theorem [11] in equation (31) and the energy of H-sequence by equation (32) in the frequency domain of equation (33):

$$E(k) = \sum_{\tau=0}^{N-1} R(\tau) W^{\tau k} = R(0) = N, \quad k = \overline{0, N-1} \tag{31}$$

$$E = \sum_{k=0}^{N-1} R(0) = N^2. \qquad (32)$$

$$E(k) = \sum_{\tau=0}^{N-1} \sum_{i=0}^{N-1} h(i)h(i+\tau)W^{\tau k} = \left|H(k)\right|^2 \qquad (33)$$

The coefficients of the DFT as complex numbers will be represented as a module and phase:

$$H(k) = (a_k + jb_k) = \sqrt{{a_k}^2 + {b_k}^2}\, e^{j\phi k} = A_k e^{j\phi k}, k = \overline{0, N-1} \qquad (34)$$

From (31) and (34), it follows that all modules $A_k = \sqrt{N}$, $k = \overline{0, N-1}$ and the sequence $\{\phi_k\} = \{arctg(b_k/a_k)\}$, $k = \overline{0, N-1}$ completely determine the phase structure of the DFT spectrum. Thus, the problem of synthesizing an H-sequence with an ideal PACF, considered in the frequency domain, is reduced to finding the rule for constructing a phase-coding sequence:

$$\{\phi_k\}, k = \overline{0, N-1} \qquad (35)$$

In general, the phase-coded sequence (35) can accept an infinite number of realizations, while leaving the amplitude DFT spectrum A_k, $k = \overline{0, N-1}$ permanent. Each time, finding the inverse DFT (IDFT) [11] is :

$$h(i) = 1/N \sum_{k=0}^{N-1} H(k)W^{-ik}, \quad k = \overline{0, N-1} \qquad (36)$$

We obtain a periodic number sequence $\{h(i)\}$, $i = \overline{0, N-1}$ with an ideal PACF, and the problem of synthesis need not be considered separately. However, with this general approach, it is very difficult to obtain a sequence with predetermined properties.

Note that in the literature, there are well-known Frank multi-phase codes of length $N = m^2$, systems of multi-phase signals, and binary (with opposite elements) [5] sequences with ideal PACF. In this case, known sequences were constructed without imposing the conditions of their effective convolution, the elements of these sequences are, as a rule, either complex or irrational numbers.

In practice, however, H-sequences with a fixed integer alphabet are of primary value. The discrete signals encoded by H-sequences with integer components are APSK signals, and phase manipulation with two states 0, π is allowed. Integer H-sequences can be obtained only in the class of real sequences, therefore the DFT spectrum must have properties 1-4 (from section III). Let us use these properties and formulate the rule for constructing the phase sequence for the generating elements of the DFT spectrum, invariant to the m-shift, as follows:

$$\left.\begin{array}{l}
\varphi_k \in 0,\ \pi \text{ for all } k \in \Psi_c \\
\textit{where} \text{ multiplicity } \Psi_c = \left\{k_n m^{n-1}, k_\nu m^{\nu-1}\right\} \\
\text{for all } k_n = \begin{cases} \overline{0, m/2}, \textit{if} \text{ m-even} \\ \overline{0, (m-1)/2}, \text{ if m-odd} \end{cases} \\
\text{and for all } \nu = \overline{1, m-1}, \\
k_\nu = \begin{cases} \overline{1, m/2}, \text{ if m-even,} \\ \overline{1, (m-1)/2}, \text{ if m-odd.} \end{cases}
\end{array}\right\} \qquad (37)$$

The frequency method of synthesizing integer H-sequences is presented in the form of the following algorithm:

Step 1: Set a uniform amplitude DFT-spectrum of the synthesized H-sequence of length $N = m^n$ in the form $k = \overline{0, N-1}$.

Step 2: Set phase structure $\{\varphi_k\}$ forming elements (37) of the total number of possible structures $S = 2^{\psi_c}$ and construct, in accordance with (25) and (27), the DFT spectrum invariant to the *m*-shift.

Step 3: Find the elements of the H-sequence $\{h(i)\}$, $i = \overline{0, N-1}$ using the IDPF (36).

Step 4: If the number of synthesized H-sequences is less than $S = 2^{\psi_c}$, then go to step 2, otherwise the synthesis is complete.

Examples of H-sequences synthesized using the presented algorithm are summarized in Table 5-9. In order to obtain a more compact record in Table 5-9, the values of only the forming elements of symmetric integer H-sequences are given, and the forming elements of inverse H-sequences are omitted.

Table 5
The Forming Elements of the H-sequences of Length $N = 16$, $m = 2$, $n = 4$

No	h(0)	h(1)	h(2)	h(4)	h(8)
1	7	1	-1	-1	-1
2	7	-1	-1	-1	-1
3	3	0	-1	-1	-1
4	3	0	1	-1	-1
5	5	1	1	-3	-3
6	5	-1	1	-3	-3
7	4	0	0	-4	-4
8	4	0	0	4	-4
9	3	0	-1	3	-5
10	3	0	-1	3	-5
11	1	1	-1	1	-3
12	1	-1	1	1	-3
13	1	0	1	1	-7
14	1	0	1	1	-7
15	0	1	0	0	16
16	16	-1	0	0	0

Table 6
The Forming Elements of the H-sequences of Length $N = 16$, $m = 4$, $n = 2$

No	h(0)	h(1)	h(2)	h(4)	h(8)
1	16	0	0	0	0
2	7	1	-1	-1	-1
3	3	0	1	-1	-1
4	5	1	1	-3	-3

5	7	-1	1	-1	-1
6	3	0	-1	-1	-1
7	5	-1	-1	-3	-3
8	8	0	0	-8	-8
9	8	0	0	8	-8
10	3	1	-1	3	-5
11	4	0	4	4	-12
12	1	1	1	1	-7
13	3	-1	-1	3	-5
14	4	0	-4	4	-12
15	1	-1	1	1	-7
16	0	0	0	0	16

Table 7
The Forming Elements of the H-sequences of Length N = 25, m = 5, n = 2

No	h(0)	h(1)	h(2)	h(5)	h(10)
1	25	0	0	0	0
2	17	2	2	-8	-8
3	23	-2	-2	-2	-2
4	15	0	0	-10	-10

Table 8
The Forming Elements of the H-sequences of Length N = 9, m = 3, n = 2

No	h(0)	h(1)	h(3)
1	9	0	0
2	3	0	-6
3	5	2	-4
4	7	-2	-2

Table 9
The Forming Elements of the H-sequences of Length N = 49, m = 7, n = 2

No	h(0)	h(1)	h(3)	h(7)	h(14)	h(21)
1	49	0	0	0	0	0
2	47	-2	-2	-2	-2	-2
3	35	0	0	0	0	0
4	37	2	2	2	-12	-12

From the analysis of the algorithm for the synthesis of H-sequences with ideal PACF (see Figure 1) and integer elements, it follows that the magnitude of the base q of the constituent elements is found as a result of the synthesis of each H-sequence.

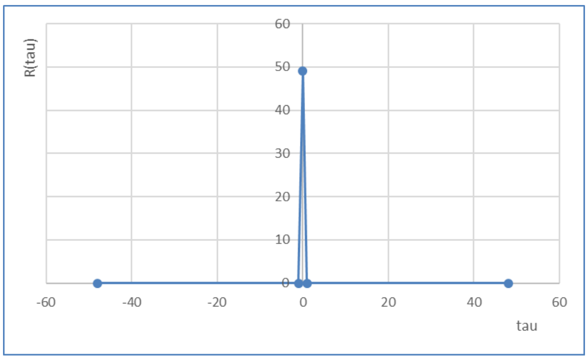

Figure 1: PACF sequence of length $N = 7^2 = 49$

For example, the value of $q \in \{2,3,4\}$ for the H-sequences from Table 5. Moreover, in a special case, the H-sequence allows building a normal system of orthogonal time-pulse-modulated (TPM) signals [13], [14].

Matched filter based on m-convolution (38) [6], [12] processes signals using $n(m-1)+1$ multiplication operations, while a cyclic convolution based filter requires m^n multiplication operations as in equation (38) for $\tau = \overline{1, N-1 = m^n-1}$:

$$R(\tau) = \sum_{i=0}^{N-1} h(i)h((i-\tau) \bmod m) = 0 \qquad (38)$$

As a result, the performance of the digital filter (with m- convolution-based signal processing) increases by $m^n / (n(m-1)+1)$ times compared to the filter that calculates cyclic convolution. Thus, for example, at N = 25, the gain will be 25/9 ≈ 2.8 times.

VI. CONCLUSION

Thus, the authors have developed a regular synthesis algorithm for the frequency domain of coding H-sequences with integer elements and ideal PACFs. In addition:

- Compared to the data from [5], the structural properties of H-sequences were defined more precisely; in particular, the boundaries of the number of degrees of freedom of symmetric H-sequences were found for arbitrary m-shift values.
- The properties of the DFT spectrum of H-sequences were studied and the rule of binary (0, π) coding of the phase spectrum for which the corresponding symmetric H-sequence was constructed.
- An upper estimate of the arithmetic calculation complexity of the DFT spectrum of H-sequences was obtained.

Each coding H-sequence allows constructing a system of orthogonal or biorthogonal cyclic APSK signals, which make it possible to perform a computationally efficient digital processing «as a whole».

REFERENCES

[1] P. K. Singya, N. Kumar, V. Bhatia and M. Alouini, "On Performance of Hexagonal, Cross, and Rectangular QAM for Multi-Relay Systems," in IEEE Access, vol. 7, pp. 60602-60616, 2019.

[2] A. V. Sadchenko O. A. Kushnirenko "QPSK-Modulation Modem Invariant to the Rotation of the Signal Constellation Plan", Electrical, Control and Communication Engineering, 2018, vol.14 No. 2, P. 149 – 156 https://doi.org/10.2478/ecce-2018-0018p.

[3] C.T. Manimegalai, R.Kumar Performance of LDPC-Coded APSK-Modulation for Wireless USB//Wseas Transactions on Communications, 2013, vol. 12, no. 12, pp. 661 -669.

[4] Liu, Y.; Chen, W.; Wang, A.; Han, C. Combination of High-Order Modulation and Non-Binary LDPC Codes over GF(7) for Non-Linear Satellite Channels. Electronics 2019, 8, 1396.

[5] M. I. Mazurkov, [Broadband radio communication systems] Nauka i tekhnika, Odessa, 2010 (in Russian), ISBN 978-966-8335-95-2.

[6] V.V. Syuzev "Synthesis methods of the fast Vilenkin-Chrestenson transforms"//Science and Education Bauman Moscow, 2014, pp159 - 187. — ISSN 1994-0408 https://doi.org/10.7463/0114.0687462

[7] M.I., Mazurkov, A.V Sokolov,. & N.A Barabanov,. "Synthesis method for bent sequences in the Vilenkin-Chrestenson basis", Radioelectron.Commun.Syst. 59, 510–517 (2016). https://doi.org/10.3103/S0735272716110054.

[8] Jordanova L., Laskov L., Dobrev D. "Constellation and Mapping Optimization of APSK Modulations used in DVB-S2"//Engineering, Technology & Applied Science Research, 2014, vol. 4, no. 5, pp. 690 - 695.

[9] Gajic, Dusan and Stankovic, Radomir, "Computation of dyadic convolution on GPU for efficient modeling of dyadic LTI systems".

Facta Universitatis - Series: Automatic Control and Robotics. 10. 59-70. 2011.

[10] Stanković, R., S., Bhattacharaya, M., Astola, J., T., "Calculation of dyadic convolution trough bynary decision diagram", Facta Universitatis - Series: Automatic Control and Robotics. Vol.8, N 1 pp.89-97. 2009. http://facta.junis.ni.ac.rs/acar/acar200901/acar2009-08.pdf

[11] Ricker D.W. "Echo Signal Processing" Business Media New York: Springer US, 2003. — XVIII, 484 p. — ISBN 978-1-4615-0312-5

[12] V. G. Labunets, S. A. Martyugin, "Fast multiparameter Walsh, Chrestenson – Vilenkin and Haar transforms", Bulletin of the South Ural State University. series"computer technologies, automatic control, radioelectronics", 2016 (4), pp.136-142.(in Russian) https://doi.org/10.14529/ctcr160416

[13] A. V. Sadchenko, O. A. Kushnirenko and A. V. Troyansky, "The Algorithm of Random Length Sequences Synthesis for Frame Synchronization of Digital Television Systems", Odes'kyi Politechnichnyi Universytet. Pratsi, no. 3 pp. 97–103, 2015. https://doi.org/10.15276/opu.3.47.2015.14

[14] A. V. Sadchenko O. A. Kushnirenko O. I. Yefimov V. V. Punko and S. Y. Parovyi "Correlation Scheme of Frame Synchronization in Communication Systems With QPSK-Modulation", Tekhnologiya i Konstruirovanie v Elektronnoi Apparature vol. 6 pp. 22–28 2017. https://doi.org/10.15222/tkea2017.6.22 (in Russian).