Angular Momentum of a Rotating Dipole

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***Abstract*—In this paper, we derive the far field electromagnetic fields of a rotating half-wave dipole antenna. Theoretically, we have demonstrated that the far electromagnetic fields of a rotating half-wave dipole carry angular momentum in the term of φ - φ′, which is absent from the stationary half-wave dipole antenna. The term $\sin(kr - \omega t + [\varphi - \varphi'])$ tells us that the electromagnetic wave propagates outward with the speed of light *c* (evidence from *k* = *ω*/*c*) from the dipole along the *r* axis and both electric and magnetic fields are spinning (oscillate with *ω* rad/s) and orbiting (rotating with ω_0 rad/s) along the *r* axis with the speed of light. The orbital frequency is evidence from the term $\varphi - \varphi' = \omega_0 t - \omega_0 t' = \omega_0 r / c$.**

***Index Terms*—angular momentum, electric field, magnetic field, rotating dipole.**

I. Introduction

An electromagnetic system radiates not only energy (linear momentum) but also angular momentum into the far zone as evidence from electrodynamics literature [1,2,3]. In this paper, we are motivated to derive the far field electromagnetic fields of a rotating half-wave dipole antenna, in order to demonstrate theoretically that such antenna would radiate angular momentum into the far zone.

II. Current distribution

A thin rod with length L, perfectly conducting, half-wave dipole antenna is located in the $x_1'x_2'$ plane and are fed at the midpoint of the rod ($\mathbf{x}_0 = \mathbf{0}$) so that the Fourier amplitude of the current distribution can be written as such

$$\mathbf{j}_\omega(x_1') = \delta(x_2')\delta(x_3') I_0 \cos(\omega x_1'/v)\hat{x}_1' \tag{1}$$

where v is the velocity of the current propagation along the rod that is equivalent to the drift velocity of the moving charges. The rod is rotating at an angular frequency ω_0 around a fixed axis ($\omega \neq \omega_0$), which is always perpendicular to the rod and passes through its midpoint.

As shown in Figure 1, we introduce a coordinate system (x_1, x_2, x_3) to describe the fixed observer (the field point) and (x_1', x_2', x_3') system to describe the source, i.e., a coordinate system that is fixed in the rod and, consequently, rotates around the x_3' axis (in fact, $x_3' = x_3$) with a constant angular frequency ω_0.

The direction of the current source in coordinate system (x_1, x_2, x_3) can be written as such

$$\hat{x}_1' = \cos(\varphi')\hat{x}_1 + \sin(\varphi')\hat{x}_2 = \mathrm{Re}\left\{(\hat{x}_1 + i\hat{x}_2)e^{-i\omega_0 t'}\right\} \tag{2}$$

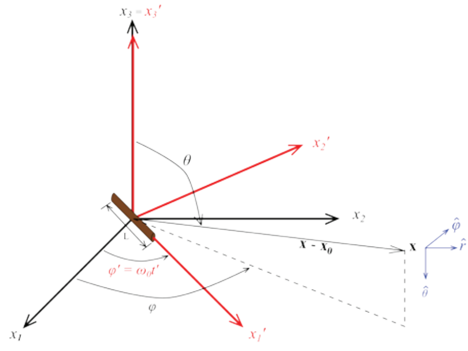

Figure 1: Geometry relevant to the current distribution formulation

where $\varphi' = \omega_0 t'$ is the azimuth angle of the source coordinate system and t' is the retarded time.

Consequently, the Fourier amplitude of the current distribution due to propagating current with angular frequency ω and speed v in the rotating rod with angular frequency ω_0 can be re-written as such

$$\mathbf{j}_\omega(x_1') = I_0 \cos(\omega x_1'/v)e^{-i\varphi'}[\hat{x}_1 + i\hat{x}_2]. \tag{3}$$

III. The magnetic field

The fields at large distances from the dipole are given in [1]. From these equations we see that

$$\mathbf{E}_\omega = c^2 \mathbf{B}_\omega \times \frac{\mathbf{k}}{\omega} \tag{4}$$

where c is the speed of light and $\mathbf{k}$ is the wave vector. Therefore it is sufficient to calculate

$$\mathbf{B}_\omega(\mathbf{x}) = -i\frac{k}{4\pi\varepsilon_0 c^2}\frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\int_{-\lambda/4}^{\lambda/4} dx_1'(\mathbf{j}_\omega e^{i\mathbf{k}\bullet(\mathbf{x}'-\mathbf{x}_0)} \times \mathbf{k}) \tag{5}$$

where $|\mathbf{x} - \mathbf{x}_0|$ (or r) is the magnitude of the difference between position vector $\mathbf{x}$ at observation point and the midpoint vector $\mathbf{x}_0$ at the source coordinate system. Further, we can estimate the magnetic field as such

$$\mathbf{B}_\omega(\mathbf{x}) = -i\frac{k}{4\pi\varepsilon_0 c^2}\frac{e^{i(kr-\varphi')}}{r} \int_{-\lambda/4}^{\lambda/4} dx_1' \left(I_0 \cos\left(\frac{\omega x_1'}{\nu}\right) e^{i\mathbf{k}\bullet\mathbf{x}_1'}[(\hat{x}_1 + i\hat{x}_2)\times \mathbf{k}] \right) \tag{6}$$

The directions of the components of the magnetic field are given by term $[(\hat{x}_1 + i\hat{x}_2)\times \mathbf{k}]$. For transformation to spherical coordinates,

$$\hat{x}_1 = \sin\theta\cos\varphi\hat{r} + \cos\theta\cos\varphi\hat{\theta} - \sin\varphi\hat{\varphi}$$
$$\hat{x}_2 = \sin\theta\sin\varphi\hat{r} + \cos\theta\sin\varphi\hat{\theta} + \cos\varphi\hat{\varphi}$$

Hence,

$$\hat{x}_1 + i\hat{x}_2 = \sin\theta(\cos\varphi + i\sin\varphi)\hat{r} + \cos\theta(\cos\varphi + i\sin\varphi)\hat{\theta} + (i\cos\varphi - \sin\varphi)\hat{\varphi}$$
$$= e^{i\varphi}(\sin\theta\hat{r} + \cos\theta\hat{\theta} + i\hat{\varphi})$$

Therefore, the directions of the components of the magnetic field are

$$(\hat{x}_1 + i\hat{x}_2)\times \mathbf{k} = ke^{i\varphi}(i\hat{\theta} - \cos\theta\hat{\varphi}) \tag{7}$$

under paraxial approximation when the wave vector $\mathbf{k} = (\omega/c)(\mathbf{r}/r)$ is propagating in the same direction as radial vector $\mathbf{r}$ with the speed of light c. Equation (6) can be re-written as such

$$\mathbf{B}_\omega(\mathbf{x}) = -iI_0\frac{k^2}{4\pi\varepsilon_0 c^2}\frac{e^{i(kr+\varphi-\varphi')}}{r}[i\hat{\theta} - \cos\theta\hat{\varphi}] \int_{-\lambda/4}^{\lambda/4} \cos\left(\frac{\omega x_1'}{\nu}\right) e^{i\mathbf{k}\bullet\mathbf{x}_1'} dx_1' \tag{8}$$

The phase error term $e^{i\mathbf{k}\bullet\mathbf{x}_1'}$ is very significant due to the fact that the length of the dipole L = λ/2 and can be estimated as such

$$\mathbf{k}\cdot\mathbf{x}_1' = \mathbf{k}\cdot(\mathbf{x}_1 + \mathbf{x}_2) = kx_1\sin\theta\cos\varphi + kx_2\sin\theta\sin\varphi.$$

Consequently

$$\mathbf{B}_\omega(\mathbf{x}) = -iI_0\frac{k^2}{4\pi\varepsilon_0 c^2}\frac{e^{i(kr+\varphi-\varphi')}}{r}[i\hat{\theta} - \cos\theta\hat{\varphi}] \int_{-\lambda/4}^{\lambda/4} \left(\cos\left(\frac{\omega x_1}{\nu}\right) e^{ik_1x_1} dx_1 + \cos\left(\frac{\omega x_2}{\nu}\right) e^{ik_2x_2} dx_2 \right) \tag{9}$$

where $k_1 = k\sin\theta\cos\varphi$ and $k_2 = k\sin\theta\sin\varphi$. Letting κ denotes either k_1 or k_2 and ζ denotes either x_1 or x_2, the solution for generic integral is given as

$$\int_{-\lambda/4}^{\lambda/4} \left(\cos\left(\frac{\omega\zeta}{\nu}\right) e^{i\kappa\zeta} d\zeta \right) = \frac{2k}{k^2 - \kappa^2}\cos\left(\kappa\frac{\lambda}{4}\right) \tag{10}$$

when condition $\nu = c$ is satisfied. Equation (12) is re-write such that

$$\mathbf{B}_\omega(\mathbf{x}) = -iI_0\frac{k}{2\pi\varepsilon_0 c^2}\frac{e^{i(kr+\varphi-\varphi')}}{r}[i\hat{\theta} - \cos\theta\hat{\varphi}] \left[\frac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\cos\varphi\right) + \frac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\sin\varphi\right) \right] \tag{11}$$

Transforming this Fourier component back to time domain and taking the (physically acceptable) real part, we obtain, in spherical coordinates

$$\mathbf{B}(t,\mathbf{x}) = \mathrm{Re}\left\{\mathbf{B}_\omega(\mathbf{x})e^{-i\omega t}\right\}$$
$$= \mathrm{Re}\left\{ I_0\frac{k}{2\pi\varepsilon_0 c^2}\frac{e^{i(kr-\omega t+\varphi-\varphi')}}{r}[\hat{\theta} + i\cos\theta\hat{\varphi}] \left[\frac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\cos\varphi\right) + \frac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\sin\varphi\right) \right] \right\}$$
$$= \mathrm{Re}\left\{ I_0\frac{k}{2\pi\varepsilon_0 c^2}\frac{\cos(kr-\omega t+\varphi-\varphi') + i\sin(kr-\omega t+\varphi-\varphi')}{r} [\hat{\theta} + i\cos\theta\hat{\varphi}] \left[\frac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\cos\varphi\right) + \frac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\sin\varphi\right) \right] \right\}$$

Let

$$\xi = \frac{I_0 k}{2\pi\varepsilon_0 c^2 r}\left[\frac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\cos\varphi\right) + \frac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\sin\varphi\right) \right]$$

Then

$$\mathbf{B}(t,\mathbf{x}) = \xi\mathrm{Re}\left\{ \left[\cos(kr-\omega t+\varphi-\varphi') + i\sin(kr-\omega t+\varphi-\varphi')\right][\hat{\theta} + i\cos\theta\hat{\varphi}] \right\}$$
$$= \xi\mathrm{Re}\left\{ \begin{array}{l} \left(\cos(kr-\omega t+\varphi-\varphi')\hat{\theta} - \sin(kr-\omega t+\varphi-\varphi')\cos\theta\hat{\varphi}\right) \\ +i\left(\sin(kr-\omega t+\varphi-\varphi')\hat{\theta} + \cos(kr-\omega t+\varphi-\varphi')\cos\theta\hat{\varphi}\right) \end{array} \right\}$$
$$= \xi[\cos(kr-\omega t+\varphi-\varphi')\hat{\theta} - \sin(kr-\omega t+\varphi-\varphi')\cos\theta\hat{\varphi}]$$

The time-varying magnetic field in the final form takes place as such

$$\mathbf{B}(t,\mathbf{x}) = \frac{\mu_0 I_0 \omega}{2\pi c r}\begin{bmatrix} \frac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\cos\varphi\right) \\ +\frac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\sin\varphi\right)\end{bmatrix} \tag{12}$$

$$\left[\cos(kr-\omega t+[\varphi-\varphi'])\hat{\theta} - \sin(kr-\omega t+[\varphi-\varphi'])\cos\theta\hat{\varphi}\right]$$

In the presence of perfect conducting ground parallel to the x_3' axis, the total magnetic field observed at the ground level is the superposition of the magnetic field obtained in Equation (12) and its image. The only component survived at ground level is the component in the direction of $\hat{\varphi}$.

Therefore, the total magnetic field is

$$\mathbf{B}(t,\mathbf{x}) = -\hat{\varphi}\frac{\mu_0 I_0 \omega}{\pi c r}\sin(kr-\omega t+[\varphi-\varphi'])\cos\theta$$

$$\begin{bmatrix} \frac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\cos\varphi\right)+ \\ \frac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\sin\varphi\right)\end{bmatrix} \tag{13}$$

The total magnetic field carries angular momentum as evidence from the azimuth term φ - φ'.

IV. THE ELECTRIC FIELD

The electric field can be obtained from Equation (4) as such

$$\mathbf{E}_\omega(\mathbf{x}) = c^2\mathbf{B}_\omega(\mathbf{x})\times\frac{\mathbf{k}}{\omega}$$

$$= -iI_0\frac{k^2}{2\pi\varepsilon_0\omega}\frac{e^{i(kr+\varphi-\varphi')}}{r}[i(\hat{\theta}\times\hat{r})-\cos\theta(\hat{\varphi}\times\hat{r})]$$

$$\begin{bmatrix} \frac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\cos\varphi\right)+ \\ \frac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\sin\varphi\right)\end{bmatrix} \tag{14}$$

$$= I_0\frac{k^2}{2\pi\varepsilon_0\omega}\frac{e^{i(kr+\varphi-\varphi')}}{r}[-\hat{\varphi}+i\cos\theta\hat{\theta}]$$

$$\begin{bmatrix} \frac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\cos\varphi\right)+ \\ \frac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\sin\varphi\right)\end{bmatrix}$$

Transforming this Fourier component back to time domain and taking the (physically acceptable) real part, we obtain, in spherical coordinates

$$\mathbf{E}(t,\mathbf{x}) = \mathrm{Re}\left\{\mathbf{E}_\omega(\mathbf{x})e^{-i\omega t}\right\}$$

$$= \mathrm{Re}\left\{\begin{matrix} I_0\frac{k^2}{2\pi\varepsilon_0\omega}\frac{e^{i(kr-\omega t+\varphi-\varphi')}}{r}[-\hat{\varphi}+i\cos\theta\hat{\theta}] \\ \begin{bmatrix} \frac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\cos\varphi\right)+ \\ \frac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\sin\varphi\right)\end{bmatrix}\end{matrix}\right\}$$

$$= \mathrm{Re}\left\{\begin{matrix} \frac{I_0 k^2}{2\pi\varepsilon_0\omega r}\cos(kr-\omega t+\varphi-\varphi')+ \\ i\sin(kr-\omega t+\varphi-\varphi')[-\hat{\varphi}+i\cos\theta\hat{\theta}] \\ \begin{bmatrix} \frac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\cos\varphi\right)+ \\ \frac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\sin\varphi\right)\end{bmatrix}\end{matrix}\right\}$$

Let

$$\chi = \frac{I_0 k^2}{2\pi\varepsilon_0\omega r}\begin{bmatrix} \frac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\cos\varphi\right)+ \\ \frac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\frac{\pi}{2}\sin\theta\sin\varphi\right)\end{bmatrix}$$

Then

$$\mathbf{E}(t,\mathbf{x}) = \chi\mathrm{Re}\left\{\begin{matrix}\cos(kr-\omega t+\varphi-\varphi')+ \\ i\sin(kr-\omega t+\varphi-\varphi')[-\hat{\varphi}+i\cos\theta\hat{\theta}]\end{matrix}\right\}$$

$$= \chi\mathrm{Re}\left\{\begin{matrix} -[\sin(kr-\omega t+\varphi-\varphi')\cos\theta\hat{\theta}+ \\ \cos(kr-\omega t+\varphi-\varphi')\hat{\varphi}]+ \\ i[\cos(kr-\omega t+\varphi-\varphi')\cos\theta\hat{\theta}- \\ \sin(kr-\omega t+\varphi-\varphi')\hat{\varphi}]\end{matrix}\right\}$$

$$= -\chi[\sin(kr-\omega t+\varphi-\varphi')\cos\theta\hat{\theta}+$$
$$\cos(kr-\omega t+\varphi-\varphi')\hat{\varphi}]$$

The electric field in the final form takes place as such

$$\mathbf{E}(t,\mathbf{x}) = -\frac{I_0 k}{2\pi\varepsilon_0 cr}\left[\begin{array}{l}\dfrac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\dfrac{\pi}{2}\sin\theta\cos\varphi\right)\\ +\dfrac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\dfrac{\pi}{2}\sin\theta\sin\varphi\right)\end{array}\right] \quad (15)$$

$$\left[\begin{array}{l}\sin(kr-\omega t+[\varphi-\varphi'])\cos\theta\hat{\theta}+\\ \cos(kr-\omega t+[\varphi-\varphi'])\hat{\varphi}\end{array}\right]$$

In the presence of perfect conducting ground parallel to the x_3' axis, the total electric field observed at the ground level is the superposition of the electric field obtained in Equation (15) and its image. The only component survived at ground level is the component in the direction of $\hat{\theta}$.

Therefore, the total electric field is

$$\mathbf{E}(t,\mathbf{x}) = -\hat{\theta}\frac{I_0 k}{\pi\varepsilon_0 cr}\sin(kr-\omega t+[\varphi-\varphi'])\cos\theta$$

$$\left[\begin{array}{l}\dfrac{1}{1-(\sin\theta\cos\varphi)^2}\cos\left(\dfrac{\pi}{2}\sin\theta\cos\varphi\right)+\\ \dfrac{1}{1-(\sin\theta\sin\varphi)^2}\cos\left(\dfrac{\pi}{2}\sin\theta\sin\varphi\right)\end{array}\right]$$

The total electric field carries angular momentum as evidence from the azimuth term φ - φ'.

V. Conclusion

Theoretically, we have demonstrated that the far electromagnetic fields of a rotating half-wave dipole carry angular momentum in the term of φ - φ′, which is absent from the stationary half-wave dipole antenna. The term $\sin(kr-\omega t+[\varphi-\varphi'])$ tells us that the electromagnetic wave propagates outward with the speed of light *c* (evidence from *k* = ω/c) from the dipole along the *r* axis and both electric and magnetic fields are spinning (oscillate with ω rad/s) and orbiting (rotating with ω_0 rad/s) along the *r* axis. The orbital frequency is evidence from term $\varphi-\varphi' = \omega_0 t-[\omega_0 t-\omega_0 r/c] = \omega_0 r/c$.

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