Comparison New Algorithm Modified Euler Based on Harmonic-Polygon Approach for Solving Ordinary Differential Equation

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***Abstract*—There are many benefits to improve Euler scheme for solving the Ordinary Differential Equation problems. Among the benefits are simple implementation and low-cost computation. However, the problem of accuracy in the Euler scheme persuades scholar to use the complex method. Therefore, the main purpose of this research is to show the development of a new modified Euler scheme that improves the accuracy of the Polygon scheme in various step sizes. The implementation of the new scheme is by using the Polygon scheme and then Harmonic mean concept that is called the Harmonic-Polygon scheme. This Harmonic-Polygon scheme can provide new advantages more than the Euler scheme could offer by solving Ordinary Differential Equation problem. Four set of problems are solved via Harmonic-Polygon. Findings showed that new scheme or Harmonic-Polygon scheme can produced much better accuracy result.**

***Index Terms*—Euler; Harmonic-Polygon Scheme; Polygon; Harmonic Mean.**

I. INTRODUCTION

Real-world problems, such as chemical reactions, weather forecasts, the population and epidemics are too difficult to be solved analytically. These problems need to be modeled using mathematical equations in order to determine the solutions. Such equation is known as the Differential Equation (DE). Based on the mathematical aspects, DE can be defined as an equation that relates the derivative of an unknown function, the function itself, the variables that define the function, as well as the functions of specific constants [1]. If the unknown function depends on one variable, the equation is called the Ordinary Differential Equation (ODE), which consists of at least one derivative coefficient: $\frac{dy}{dx}, \frac{d^2 y}{dx^2}$.

This study has implemented the Euler scheme to solve the ODE problems. Euler's method is also called a tangent line method and is the simplest numerical method for solving ODE. This method was developed by Leonhard Euler in 1768 and it is suitable for quick programming, simple implementation and low-cost computational [2]. However, the accuracy factor persuades researchers to use another complex method to replace Euler method. The primary aim of this investigation is to discover a new solution as accurate as possible with the exact solution using the mean concept. Thus, the improvements in the current Euler method are applied to the concept of the Modified Euler.

One of the techniques to improve the performance of the Euler scheme is to modify the original scheme [2][3][4]. The modification method was focused on finding the average gradients. This study has proposed a new scheme using the concept of harmonic mean to improve the Euler method.

After that, the scheme will be analyzed using a mathematical software. In this research, the mathematical software used is Scilab 5.4.1 Programming. Before using the mathematical software, the algorithm must be done. The mathematical software and algorithm development is closely linked to the problem represented by a mathematical model. Algorithm is an order of instructions to solve problems rationally in simple language [5]. Algorithm also can be demonstrated in stage by stage to solve the problems. Algorithm naturally is conceptual or abstract. Therefore, the researcher needs a way to delegate the tasks that can be communicated to humans or computers, which is called pseudo code.

Algorithms describe the elements involved clearly and then convert the algorithms into the program code, which is done more easily in a programming language. According to [6], construction process in mathematical software includes as follows.

i. The design and analysis algorithms
ii. Algorithm coding
iii. Testing
iv. Details documentation
v. Distribution and maintenance of the software

Once the algorithm is developed, a computer program is implemented to test the effectiveness of the algorithm. The code program is written using Scilab 5.4.1 Programming. At the final stage, Polygon and Harmonic-Polygon Scheme will be compared with the exact solution.

II. CONSTRUCTION OF HARMONIC POLYGON SCHEME

This section discusses about the implementation of Harmonic mean in the Polygon scheme. The authors gained the modified Euler scheme used by [3] and [7] in the process to develop the proposed scheme. Those researchers used arithmetic mean concept within two coordinate points of function to improve their method. The technique of the improved Euler Scheme is called the modified Euler Scheme, also known as Polygon.

The method proposed by the authors are used from Euler method, which is similar to Equation (1).

$$y_{i+1} = yi + \Delta t f(t_0, y_0) \quad (1)$$

The equation is modified by using the concept of average. The proposed average is Harmonic mean of the two midpoints is written in Equation (2).

$$\frac{2x_0, x_1}{x_0 + x_1}, \frac{2y_0, y_1}{y_0 + y_1} \quad (2)$$

Equation (2) also can be written as Equation (3).

$$\frac{2x_0(x_0 + h)}{x_0 + (x_0 + h)}, \frac{2y_0(y_0 + hf(x_0, y_0))}{y_0 + (y_0 + hf(x_0, y_0))} \quad (3)$$

If an equation through a point, such as P (x_0, y_0), with the gradient through the Equation (3), a new equation can be generated as given by Equation (4).

$$\frac{y - y_0}{h} = f\left(\frac{2x_0(x_0 + h)}{x_0 + (x_0 + h)}, \frac{2y_0(y_0 + hf(x_0, y_0))}{y_0 + (y_0 + hf(x_0, y_0))}\right)$$
$$y - y_0 = hf\left(\frac{2x_0(x_0 + h)}{x_0 + (x_0 + h)}, \frac{2y_0(y_0 + hf(x_0, y_0))}{y_0 + (y_0 + hf(x_0, y_0))}\right) \quad (4)$$
$$y = y_0 + hf\left(\frac{2x_0(x_0 + h)}{x_0 + (x_0 + h)}, \frac{2y_0(y_0 + hf(x_0, y_0))}{y_0 + (y_0 + hf(x_0, y_0))}\right)$$

Euler can be more stable and accurate modification is made by using the slope of the function at the estimated midpoints of (x_0, y_0) and (x_1, y_2) to approximate y_{i+1}. This research used the equation (1) and equation (3) to produce equation (4) is called Harmonic-Polygon (HP). HP scheme can be written as equation (5).

$$y_{i+1} = y_0 + hf\left(\frac{2x_0(x_0 + h)}{x_0 + (x_0 + h)}, \frac{2y_0(y_0 + hf(x_0, y_0))}{y_0 + (y_0 + hf(x_0, y_0))}\right) \quad (5)$$

III. DEVELOPMENT ALGORITHM MODIFIED EULER

In this section, two algorithms of modified Euler are compared with the exact solution. The algorithms are proposed by [8] and the authors. The purpose of this research to solve the Ordinary Differential Equation over the interval from x = 0 to 20 for Problem 1, 2 and 3 using a step size of 0.001, 0.01 and 0.01. Consider the equation $y' = -2y+2$ with exact solution which is $y(x) = 1-2e^{(-2x)}$. Time was recorded to compare each algorithm that gave the closest answer to the exact solution. Equation (6) was proposed by [8] and the author proposed Equation (5). The Equation (6) is as follows.

$$y_{i+1} = y_i + hf[x_i + \frac{h}{2}, y_i + \frac{h}{2}\left(\frac{f(x_i, y_i + f(x_{i+1}, y_i + h(x_i, y_i))}{2}\right)] \quad (6)$$

A. Polygon Algorithm

1. Start
2. Set x_0, y_0, h, x_i value.
3. Calculate value of $n = \frac{x_n - x_0}{h}$
4. Start processing time
5. Calculate $x_{i+1} = x_i + h$
6. Problem Equation $y' = -2y+2$
7. If $(i \le n)$ start_if
   - 7.1 Calculate exact solution $a(i)$, $y(x) = 1-2e^{(-2x)}$
   - 7.2 Calculate latest value of $y(i)$
     - a. Set A ← $f(x_i, y_i) + f(x_{i+1}, y_{i+1})$
     - b. Set B ← A/2
     - c. Set C ← $f = x_i + 2\, h/2, y_i + h/2 * B$
     - d. $y_{i+1} \leftarrow y_i + h * C$
   - 7.3 Calculate maximum error $|E_x - E_v|$

   End If
8. Print Maximum Error
9. End processing time.
10. Print processing time, *y* value.
11. End

B. Harmonic Polygon Algorithm

1. Start
2. Set x_0, y_0, h, x_i value.
3. Calculate value of $n = \frac{x_n - x_0}{h}$
4. Start processing time
5. Calculate $x_{i+1} = x_i + h$
6. Problem Equation $y' = -2y+2$
7. If $(i \le n)$ start_if
   - 7.1 Calculate exact solution $a(i)$, $y(x) = 1-2e^{(-2x)}$
   - 7.2 Calculate latest value of $y(i)$
     - a. Set A ← $f\left(\frac{2x_i + x_{i+1}}{x_i + x_{i+1}}, \frac{2y_i + y_{i+1}}{y_i + y_{i+1}}\right)$
     - b. Set B ← $y_{i+1} \leftarrow y_i + h * C$
   - 7.3 Calculate maximum error $|E_x - E_v|$

   End If
8. Print Maximum Error
9. End processing time.
10. Print processing time, *y* value.
11. End

IV. NUMERICAL RESULT

This section discusses the results of the four set of first order Ordinary Differential Equations (ODE). The Harmonic-Polygon (HP) scheme is tested with four set of ODE problems. Three different step sizes which are 0.001, 0.01 and 0.1 are used during testing. The various step sizes will give impact to the solution using this scheme. The schemes are transferred into algorithm before being tested.

Then, the algorithms are transferred into Scilab 5.4.1 programming to test the efficiency of the schemes. The measurement of efficiency refers to a minimum of maximum error after a complete cycle. The results of the HP scheme are compared with the P scheme to simplify the effectiveness of the scheme. Table 1 refers to a set of problem with exact solution in first order ODE.

Table 2 displays the results of the problems. The results shown in Table 2 are the comparison of the exact value for the two modified Euler which is P and HP scheme. The error involved in this case is called relative error and can be calculated [10] as Error= $|E_x - E_v|$.

$$Ex = Exact\_value,\ Ev = Euler's\_modified\_value$$

Table 1
Set of Problem First Order ODE

Equation	Exact Solution	Initial Value	Interval of Integration	Source
$y' = -2y + 2$	$y(x) = 1 - 2e^{-2x}$	$y(0) = -1$	$0 \le x \le 20$	[4]
$y' = (y - 1)$	$y(x) = \frac{0.5}{(1-0.5)e^{-(x+0.5)}}$	$y(0) = 0.5$	$0 \le x \le 20$	[9]
$y' = 5y(y - 1)$	$y(x) = \frac{0.5}{(1-0.5)e^{-5(x+0.5)}}$	$y(0) = 1$	$0 \le x \le 20$	[9]
$y' = y \cos x$	$y(x) = e^{\sin x}$	$y(0) = 1$	$0 \le x \le 10$	[8]

Table 2
Result for Ordinary Differential Problems

Scheme	Polygon (P)			Harmonic-Polygon (HP)		
Step Size	*h=0.001*	*h=0.01*	*h=0.1*	*h=0.001*	*h=0.01*	*h=0.1*
Problem 1	4.91E-07	5.00E-05	5.72E-03	5.55E-14	5.55E-15	4.44E-16
Problem 2	7.29E-09	1.00E-06	7.70E-05	5.35E-09	1.00E-06	5.60E-05
Problem 3	1.83E-07	1.90E-05	2.39E-03	1.34E-07	1.40E-05	1.72E-03
Problem 4	3.57E-03	3.56E-02	3.39E-01	3.57E-03	3.55E-02	3.34E-01

V. Discussion

Upon observation, the Harmonic-Polygon (HP) scheme fit the troubleshooting results for Problem 1 because it gave values that are more accurate. For example, at 0.001 step size, the Polygon (P) scheme gave the maximum error of 4.91E-07, compared to the HP scheme, which gave the maximum error of 5.55E-14. There was a 100% increase in the accuracy of the performance of the HP scheme compared to the P scheme. Similarly, at the step size of 0.01, the P scheme gave the maximum error of 5.00E-05, while the HP scheme gave the maximum error of 5.55E-15. Finally, after the step size was increased to 0.1, the HP scheme had given the smaller maximum error value than the P scheme at 4.44E-16.

Meanwhile, Problem 2 was a nonlinear ODE problem. The result shows that the HP scheme has provided a more accurate performance compared to the P scheme. At a small step size, h = 0.001, the HP scheme gave a smaller error of 5.35E-09 compared to the P scheme's error of 7.29E-09. Meanwhile, at a larger step size, h = 0.1, the maximum error for the HP scheme was 5.60E-05, which was smaller than the value from the P scheme at 7.70E-05. These results clearly indicated that there was a 27.27% improvement in the performance.

The results for Problem 3, which is also a nonlinear ODE. These results clearly showed that the proposed HP scheme was more suitable. This is because the HP scheme gave performances that are more accurate at each step size. At step size of h = 0.001, the maximum error for the HP scheme was 1.34E-04, compared to the maximum error for the P scheme, which was 1.83E-07. When the step size was h = 0.01, the error value for the P scheme was 1.90E-05, while the HP scheme gave the error value of 1.40E-05. At the biggest step size, the P scheme gave the error value of 2.39E-03, while the HP scheme gave the maximum error of only 1.72E-03.

VI. Conclusion

This research proposed a new scheme using the modified Euler method that is called HP scheme as the finding of this research. Subsequently, the HP scheme is compared with the P scheme. Each scheme is tested by using Scilab 5.4.1 Programming and compared with the exact solution. Usually, the ordinary Euler method using a small step size gives a result that is almost to the exact solution. However, the HP scheme's result is also close to the exact solution while using a bigger step size (h = 0.1). The benefits of using larger step size will reduce the complexity step and processing time. As a conclusion, the HP scheme can be an alternative algorithm to solve ODE problems.

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