Population Game Model for Epidemic Dynamics with Two Classes of Vaccine-induced Immunity

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***Abstract*–Behavioural factors play a key and pivotal role in the success of a voluntary vaccination programme for combating infectious diseases. Individuals usually base their voluntary vaccination decisions on the perceived costs of vaccination and infection. The perceived cost of vaccination is easily influenced by the degree of protection conferred by vaccines against infection, also known as vaccine efficacy. Although certain vaccines have a decrease in its effectiveness in specific duration of time, they do offer a reduction of transmissibility and faster recovery for vaccinated infected individuals. These additional characteristics of imperfect vaccines are well-captured in an epidemic model with two classes of vaccine-induced immunity. In this paper, the interplays between these characteristics of vaccines, the dynamics of vaccination uptake and epidemics are investigated in the vaccination population games framework. Specifically, we study to what extent the population- and individual-level vaccination rates are influenced by these characteristics of vaccines at equilibrium state.**

***Index Terms*–Two Classes of Vaccine-Induced Immunity; Vaccination Population Games; Vaccine Efficacy.**

I. INTRODUCTION

In modelling the voluntary vaccination behaviour on the disease spread by using game-theoretical approach (see review paper [1]), at microscopic level, upon receiving information on the diseases, susceptible individuals (i.e. the players of the game) decide whether or not to take vaccination (i.e. choosing vaccination or non-vaccination strategy) based on perceived costs of vaccination and disease. Individuals aim at minimizing the cost (i.e. maximizing the payoff). By opting for vaccination strategy, it is generally assumed that individuals get vaccinated immediately and the vaccine provides immunity against infection risks completely. However, if the assumption of perfect vaccine is relaxed, the perceived cost of vaccination could be vastly altered by the degree of protection offered by vaccines, which is highly associated with the vaccine efficacy and vaccine failure.

The vaccine efficacy is referred to as the theoretical success rate in preventing vaccinated individuals from becoming infected with the disease [2]. In reviewing the literature, it is not uncommon that the assumptions about perfect vaccine efficacy are further relaxed so as to better reflecting the complexity of epidemic dynamics with vaccination. When vaccine is imperfect, the critical vaccination threshold for disease eradication becomes higher [3] due to the fact that being vaccinated does not necessarily confer vaccine-induced immunity. In general, vaccine failure could be categorized into the following three types. First, vaccine failure in take ("all-or-nothing") which means that vaccine may not be able to generate immunity in a portion of people vaccinated [4]. Second, vaccine may not only offer partial protection to vaccinated individuals by lowering individuals' susceptibility to infection, but also reduce the subsequent transmissibility and speed up recovery if the vaccinated individuals suffer from infection (i.e. breakthrough infection). This type of vaccine could be described as "leaky" in [4-5] or formally grouped as vaccine failure in degree. Centers for Disease Control and Prevention (CDC) [6] claims that if a person vaccinated with chickenpox vaccine does catch the disease, it is typically not that serious compared to other non-vaccinators. Also, he/she may be able to recover faster. Third, the vaccine- and disease-induced immunity for quite a number of diseases fade with time, i.e. vaccine failure in duration [4,7]. Take the pertussis (whooping cough) vaccine as an example. For adolescents and adults, the vaccine could provide protection to roughly 7 out of 10 people in their first year after receiving the vaccine, whereas it just protects 3 or 4 out of 10 people perfectly in four years after being vaccinated [8].

In voluntary vaccination program, individuals do not usually get vaccinated simultaneously. It follows that the population may consist of vaccinated individuals with fully protective vaccine-induced immunity, together with those who only have partial protection. This coexistence of fully and partially protected vaccinated individuals as well as its epidemic dynamics are well-captured in a Susceptible-Vaccinated-Infected-Recovered-Susceptible (SVIRS) epidemic model with two classes of vaccine-induced immunity [9], in which vaccinated individuals first acquire high vaccine-induced immunity with full protection from the disease. Then, their immunity wanes in two stages, namely from high to low immunity (i.e. individuals still have some partial protection) and from low to no immunity. In addition, the extended model in [9] also assumes that in breakthrough infections, the vaccine may be able to reduce transmissibility and speed up recovery for vaccinated infected individuals. Hence, the model is particularly useful for studying the epidemic dynamics with vaccine failure in degree and in duration.

It could be postulated that people would not choose to vaccinate until the vaccine was sufficiently efficient in protecting vaccinated individuals from being infected, moreover, an increase in vaccine efficacy would boost the

vaccination rates. Having said that, when rational individuals act in their self-interest, free-riding effects can cause the vaccine uptake drops when the vaccine efficacy is high [10]. For vaccine immunity with low waning rate, vaccine coverage level is usually low but stable [11]. The reason is that the longer duration of protection given by the vaccine, the lower awareness of infection risk among susceptibles will be, and this usually cause severe infrequent epidemics [12]. The probability of non-vaccinating increases when vaccine is imperfect and this pose extra burden to the overall cost of optimum vaccination strategy [13]. It is found that the disease may invade in scale-free networks as vaccination behaviour is hard to spread across the population whenever the vaccination cost exceeds its threshold value for the vaccine imperfection [14]. Another likely consequence of the imperfect vaccine on voluntary vaccination behaviour is the phenomenon of multiple equilibria vaccination rates [15].

In light of the above-mentioned literatures, there exists an important interplay between vaccine imperfection, vaccination coverage and disease dynamics. But, these findings could be further elaborated by adding the definition of vaccine efficacy such that they are not simply limited to the reduction of susceptibility for vaccinated individuals. However, the influence of some other characteristics of imperfect vaccines (namely, the transition rate from high to low immunity for vaccinated individuals, the reduction of transmissibility and faster recovery in breakthrough infection) on individual vaccination decision-making has received relatively few attentions.

Hence, in this paper, we make use of the SVIRS epidemic model with a two-class vaccine-induced immunity and additional characteristics of imperfect vaccine in [9], along with the vaccination population games framework [15], to explore the influence of these characteristics of vaccine failure in degree on the individual and population vaccination rates as well as the consequent effects on epidemic dynamics, without focussing on a specific vaccine-preventable disease. Besides that, the set of parameter values is purposely chosen to illustrate certain specific scenario or principle and explain the dynamical behaviour which the model can exhibit.

In Section II, we present the details of vaccination population games for two-class vaccine-induced immunity model. Results are discussed in Section III and conclusion is given in Section IV.

II. Model Formulation

We develop the continuous-vaccination population game model for epidemic SVIRS dynamics (without demography) with two-class vaccine-induced immunity by taking into account the three additional characteristics of imperfect vaccines in the following four subsections.

A. Population-scale dynamics

The population-scale dynamics is given below:

$$\begin{aligned}\dot{S} &= -\lambda S - \bar{\pi} S + \omega_R R + \omega_V V_2 \\ \dot{I} &= \lambda S - \gamma_u I \\ \dot{W} &= \sigma\lambda V_2 - \gamma_v W \\ \dot{R} &= \gamma_u I + \gamma_v W - \omega_R R \\ \dot{V}_1 &= \bar{\pi} S - \gamma_1 V_1 \\ \dot{V}_2 &= \gamma_1 V_1 - \sigma\lambda V_2 - \omega_V V_2\end{aligned} \tag{1}$$

where dot represents time derivative. All variables and parameters in the model are listed in Table 1. The total population at time t, $N(t)$, is divided into six compartments (or classes, states). We specifically denote the vaccination rate, $\bar{\pi}$, with bar notation to emphasize that the quantity is of average population rate. The susceptibles, S, shift to the unvaccinated infected class, I, at the rate λ. We assume that individuals in the V_1 class are fully protected from infection with high vaccine-induced immunity until they progress to the V_2 class at a rate γ_1, in which they only have partial protection with low vaccine-induced immunity. When individuals in the V_2 class suffer the breakthrough infection, they transfer to the vaccinated infected class, W, at a rate of $\sigma\lambda$ where $\sigma \in (0,1]$ is the probability of vaccine failure. Vaccinated infected individuals have faster recovery than unvaccinated infected individuals (i.e. $\gamma_u < \gamma_v$). We assume that individuals in W class have reduction of transmissibility, $\theta \in (0,1]$. Hence, the force of infection is given by $\lambda = \beta \frac{I+\theta W}{N}$.

Table 1
Description of the variables and parameters of the model (1)

Variables	Description
S	Susceptible individuals
I	Unvaccinated and infected individuals
W	Vaccinated and infected individuals
R	Recovered individuals
V_1	Vaccinated individuals with high vaccine-induced immunity
V_2	Vaccinated individuals with low vaccine-induced immunity
Parameters	**Description**
λ	Force of infection
β	Disease transmission rate
θ	Reduction of transmissibility for individuals in W class
$\bar{\pi}$	Population vaccination rate
σ	Reduction of susceptibility for vaccinated individuals in V_2 class; or equivalently, the probability of vaccine failure in degree, where $1-\sigma$ gives vaccine efficacy
ω_R	Disease-induced immunity waning rate
ω_V	Vaccine-induced immunity waning rate
γ_u	Recovery rate for unvaccinated infected individuals
γ_v	Recovery rate for vaccinated infected individuals
γ_1	Transition rate for individuals in V_1 class to V_2 class

The disease-free equilibrium (DFE) of model (1) is given by $E_0 = (S_0, I_0, W_0, R_0, V_{10}, V_{20})$, where $I_0 = W_0 = R_0 = 0$, $S_0 = \frac{\omega_V \gamma_1}{\omega_V \gamma_1 + (\omega_V + \gamma_1)\bar{\pi}}$, $V_{10} = \frac{\omega_V \bar{\pi}}{\omega_V \gamma_1 + (\omega_V + \gamma_1)\bar{\pi}}$, $V_{20} = \frac{\gamma_1 \bar{\pi}}{\omega_V \gamma_1 + (\omega_V + \gamma_1)\bar{\pi}}$ and its effective reproduction number is:

$$R_{\text{vac}} = \frac{\beta}{\gamma_u \gamma_v}\left[\frac{\gamma_v \omega_V \gamma_1 + \sigma\theta\gamma_1\bar{\pi}\gamma_u}{\omega_V \gamma_1 + (\omega_V + \gamma_1)\bar{\pi}}\right] \tag{2}$$

Let $E^* = (S^*, I^*, W^*, R^*, V_1^*, V_2^*)$ denote any endemic equilibrium point (EEP) of model (1). The non-zero equilibria of the model satisfy the following quadratic equation:

$$a_2(\lambda^*)^2 + a_1\lambda^* + a_0 = 0 \tag{3}$$

where: $a_2 = \sigma(\omega_R + \gamma_u)\gamma_1\gamma_v > 0$

$$\begin{aligned} a_1 &= \sigma\gamma_u\left[\gamma_1\bar{\pi}(\omega_R + \gamma_v) + \gamma_v\omega_R(\gamma_1 + \bar{\pi})\right] \\ &\quad + \gamma_1\gamma_v\left[\omega_V(\omega_R + \gamma_u) - \beta\sigma\omega_R\right] \\ a_0 &= \omega_R\gamma_u\gamma_v[\omega_V\gamma_1 + (\omega_V + \gamma_1)\bar{\pi}](1 - R_{\text{vac}}) \end{aligned}$$

The quadratic Equation (3) admits at most two feasible solutions.

In the absence of vaccination (i.e. $\bar{\pi} = 0$), solving the quadratic Equation (3) gives two solutions, where one of them, $\lambda^* = -\frac{\omega_V}{\sigma} < 0$, is biologically infeasible. Thus, we take:

$$\lambda^*(\bar{\pi} = 0) = (\beta - \gamma_u)\frac{\omega_R}{\omega_R + \gamma_u} \tag{4}$$

In the case whereby susceptibles take vaccination instantly (i.e. $\bar{\pi} \to \infty$), by rearranging equation (3) and finding the limit using L'Hopital's rule, we obtain:

$$\lim_{\bar{\pi}\to\infty} \lambda^* = \frac{[\sigma\theta\beta\,\gamma_1 - \gamma_v(\gamma_1 + \omega_V)]\omega_R}{\sigma[\gamma_1(\omega_R + \gamma_v) + \gamma_v\omega_R]} \tag{5}$$

B. Individual-scale Dynamics

An efficient individual-scale model is developed based on a Markov process with variable transition rates derived from the population-scale model. That is, when the population dynamics reach its equilibrium (denoted by asterisk in superscript), the changes in a single individual's disease state are governed by the following continuous-time Markov process [17]:

$$\dot{\mathbf{x}}(t) = \mathbf{Q}^*\mathbf{x}(t)$$

with initial condition $\mathbf{x}(0) = [1, 0, 0, 0, 0, 0]^T$, where $\mathbf{x}(t) = [S(t), I(t), W(t), R(t), V_1(t), V_2(t)]^T$ and:

$$\mathbf{Q}^* = \begin{bmatrix} -\lambda^* - \pi & 0 & 0 & \omega_R & 0 & \omega_V \\ \lambda^* & -\gamma_u & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_v & 0 & 0 & \sigma\lambda^* \\ 0 & \gamma_u & \gamma_v & -\omega_R & 0 & 0 \\ \pi & 0 & 0 & 0 & -\gamma_1 & 0 \\ 0 & 0 & 0 & 0 & \gamma_1 & -\sigma\lambda^* - \omega_V \end{bmatrix} \tag{6}$$

In individual-scale dynamics, we denote the individual vaccination strategy rate with π (without bar notation). Note that π could either be the same or different from the population vaccination rate, $\bar{\pi}$. In the context of game theory, the existence of a (Nash) equilibrium in a population game means a stable collection of individual strategies such that no one has any incentive to unilaterally switch his/her strategy [13]. Since player is regarded as playing the game against a single representative "individual" who plays the population average strategy in population games [16], the individual vaccination rate, π, is said to be the same as the population vaccination rate, $\bar{\pi}$, at Nash equilibrium, $\pi^* = \pi = \bar{\pi}$. Also, at Nash equilibrium, each player in a game is assumed to have selected the best response to the population's strategy.

C. Utility Calculation

In population games, the utility (i.e. a measure of preference) of any strategy is dependent on both the individual's strategy and population average strategy. Based on the assumption that nearly all individuals in the population use the population average strategy, the population is so large that its epidemic dynamics is not dramatically affected by the change of a single individual's vaccination strategy [17].

Following [15], the closed form of the expected utility is given by:

$$U(\pi, \bar{\pi}) = [\mathbf{f}^T + \mathbf{1}^T(\mathbf{F} \bullet \mathbf{Q}^*)](h\mathbf{I} - \mathbf{Q}^*)^{-1}\mathbf{x}(0) \tag{7}$$

where h represents the discount rate, $\mathbf{1} = [1, 1, 1, 1, 1, 1]^T$, $\mathbf{I}$ denotes identity matrix, $\mathbf{f}$ is the vector of utility gains per unit time for individuals of each class and $\mathbf{F}$ gives the vector of instantaneous utility gains corresponds to each transition of state. $\mathbf{F} \bullet \mathbf{Q}^*$ represents the Hadamard product. That is, the product of the components of $\mathbf{F}$ and $\mathbf{Q}^*$.

When a person stays in the unvaccinated infected (resp. vaccinated infected) class, he/she accumulates the infection cost, c_I (resp. c_W). An instantaneous vaccination cost c_V is being incurred to susceptibles when they move to vaccinated state in the V_1 class. The c_V includes not only the monetary cost (e.g. time spent) of getting vaccination, but also the psychological burden of developing vaccine side effects (VSE). Thus:

$$\mathbf{f} = \begin{bmatrix} 0 \\ -c_I \\ -c_W \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c_V & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By using Equation (7) and taking $\lim_{h\to 0} hU(\pi, \bar{\pi})$, the utility of strategy π to an individual in a population at steady state with strategy $\bar{\pi}$ becomes:

$$U(\pi, \bar{\pi}) = \frac{-\omega_R\gamma_1[u_1\lambda^* + u_2]}{u_3(\lambda^*)^2 + u_4\lambda^* + u_5} \tag{8}$$

where: $u_1 = (\sigma\lambda^* + \omega_V)\gamma_v c_I + \pi\sigma\gamma_u c_W$

$u_2 = \gamma_u\gamma_v\pi\,(\sigma\lambda^* + \omega_V)c_V$

$u_3 = \sigma(\omega_R + \gamma_u)\gamma_1\gamma_v$

$u_4 = \gamma_1\gamma_v[\omega_V(\omega_R + \gamma_u) + \gamma_u\omega_R\sigma] + \sigma\pi\gamma_u[\omega_R(\gamma_1 + \gamma_v) + \gamma_1\gamma_v]$

$u_5 = \gamma_u\gamma_v\omega_R[\omega_V\gamma_1 + (\omega_V + \gamma_1)\pi]$

As all the epidemiological parameters are positive, the utility calculated by Equation (8) is always in negative value (also known as disutility). Assuming that the population vaccination rate $\bar{\pi}$ is given, individuals aim to minimize the loss of utility (i.e. maximize the disutility) by choosing their own individual vaccination rate π. We assume that $c_V \le c_I$ and write the relative cost of vaccination to cost of infection as $c = \frac{c_V}{c_I}$, where $0 \le c \le 1$. For simplicity, hereinafter, we set $\omega_R = \omega_V = \omega$ and $c_W = c_I = 1$.

We then compute the rate of change in utility when the individual vaccination rate π is varied by differentiating the utility Equation (8) with respect to π. By equating the resulting derivative to zero, we obtain the following critical value c:

$$c = \frac{\lambda^*[\sigma^2(\lambda^*)^2 c_{11} + \sigma\omega\lambda^* c_{12} + \omega^2 c_{13}]}{\sigma^2(\lambda^*)^3 c_{21} + \sigma\omega\,(\lambda^*)^2 c_{22} + \omega^2\lambda^* c_{23} + \omega^3 c_{24}} \tag{9}$$

where: $c_{11} = \gamma_v(\gamma_1 + \omega) - \gamma_1\gamma_u$

$c_{12} = 2\gamma_v(\gamma_1 + \omega) - (\sigma + 1)\gamma_1\gamma_u$

$$c_{13} = \gamma_v(\gamma_1 + \omega) - \sigma\gamma_1\gamma_u$$
$$c_{21} = \gamma_1\gamma_v(\gamma_u + \omega)$$
$$c_{22} = \gamma_1\gamma_v[\gamma_u(\sigma + 2) + 2\omega]$$
$$c_{23} = \gamma_1\gamma_v(\omega + 2\sigma\gamma_u + \gamma_u)$$
$$c_{24} = \gamma_1\gamma_u\gamma_v$$

Rearranging Equation (9), we obtain the following cubic equation in terms of λ^*:

$$A_3(\lambda^*)^3 + A_2(\lambda^*)^2 + A_1\lambda^* + A_0 = 0 \quad (10)$$

where: $A_3 = \sigma^2[\gamma_1\gamma_v(\gamma_u + \omega)c + \gamma_1\gamma_u - \gamma_v(\gamma_1 + \omega)]$
$A_2 = \sigma\omega\{\gamma_1\gamma_v[\gamma_u(\sigma + 2) + 2\omega]c + \gamma_1\gamma_u(\sigma + 1) - 2\gamma_v(\gamma_1 + \omega)\}$
$A_1 = \omega^2[\gamma_1\gamma_v(\omega + 2\sigma\gamma_u + \gamma_u)c + \sigma\gamma_1\gamma_u - \gamma_v(\gamma_1 + \omega)]$
$A_0 = \omega^3\gamma_1\gamma_u\gamma_v c > 0$

D. Population Games Analysis

We assume that individuals are fully rational in making their vaccination decision and have complete knowledge of the epidemiological parameters, which includes the three additional parameters in the two-class vaccine-induced immunity model, namely the duration of staying in V_1 class after vaccination, γ_1, the recovery rate, γ_v and the reduced transmissibility, θ, for vaccinated infected individuals.

Since Equation (3) is not linear and the mathematical relation between $\bar{\pi}$ and λ^* is not necessarily one-to-one, we could not replace the terms λ^* in Equation (9) explicitly with its corresponding $\bar{\pi}$ in order to examine the individuals' best response (of their own vaccination rate π) on the population vaccination rate $\bar{\pi}$. Considering that, we define the individual best response correspondence, π_{best}, by subdividing the relative cost of vaccination, c, into the following three subintervals:

$$\pi_{\text{best}}(c) = \begin{cases} 0 & \text{if} \quad c > c_{\text{no}} \\ [0, \infty) & \text{if } c_{\text{inst}} \le c \le c_{\text{no}} \\ \infty & \text{if} \quad c < c_{\text{inst}} \end{cases} \quad (11)$$

By taking $\omega_R = \omega_V = \omega$, we first substitute Equation (4) into Equation (9) so as to determine the critical value of the relative cost of vaccination to infection, c, for the zero vaccination rate (i.e. $\pi^* = \pi = \bar{\pi} = 0$), that is, the cost threshold for no vaccination, c_{no}. If:

$$c > c_{\text{no}} = \frac{(\beta - \gamma_u)[(\gamma_1 + \omega)k_0 - \sigma\gamma_1\gamma_u(\beta + \omega)]}{\beta\gamma_1(\gamma_u + \omega)k_0} \quad (12)$$

where $k_0 = \gamma_v[(1 - \sigma)\gamma_u + \sigma\beta + \omega]$, then no one in the population will vaccinate at Nash equilibrium. Similarly, by substituting Equation (5) into Equation (9), we conclude that if $\sigma\theta\beta\gamma_1 > \gamma_v(\gamma_1 + \omega)$ and $c < c_{\text{inst}}$, with c_{inst} is given by:

$$c_{\text{inst}} = \frac{(\gamma_1 k_1 - k_4\gamma_v)[k_1(k_4\gamma_v - \gamma_1\gamma_u) - k_3\sigma\gamma_u + \gamma_v k_4 k_5]}{\gamma_1\gamma_v\{k_5\gamma_1(k_1)^2 + k_1(k_3\sigma\gamma_u - k_5 k_2) + \gamma_u\sigma\omega k_3 - \gamma_v\omega k_4 k_5\}} \quad (13)$$

where: $k_1 = \theta\sigma\beta$
$k_2 = \gamma_v(\gamma_1 + \omega) - \gamma_1\omega$
$k_3 = \omega(\gamma_1 + \gamma_v) + \gamma_v\gamma_1$
$k_4 = \gamma_1 + \omega$
$k_5 = \gamma_u + \omega$

then the susceptibles will vaccinate instantly (i.e. $\pi^* = \pi = \bar{\pi} \to \infty$) at Nash equilibrium. Whenever $c_{\text{inst}} \le c \le c_{\text{no}}$, the (Nash) vaccination rate is finite (i.e. $\pi^* = \pi = \bar{\pi} \in [0, \infty)$).

As for every cubic equation with real coefficients, there always exists at least one solution among the real numbers. It is easy to obtain the closed-form discriminant of the cubic equation (10) with a formula:

$$\Delta = 18A_3A_2A_1A_0 - 4A_2^3A_0 + A_2^2A_1^2 - 4A_3A_1^3 - 27A_3^2A_0^2 \quad (14)$$

By taking $\Delta = 0$, the cubic equation (10) is said to have a multiple root in which all its roots are real. This implies that the fold (or saddle-node) bifurcation occurs in the utility function (8). That is, the multiple endemic equilibria λ^* (and its corresponding population vaccination rates $\bar{\pi}$, if exist) collide and merge into one. After some algebraic manipulations, we obtain the following quadratic equation in terms of c, which gives the location of fold bifurcation:

$$(k_7)^2\{(b_2)^2c^2 + 2\gamma_1\gamma_v b_1 c + (b_0)^2\} = 0 \quad (15)$$

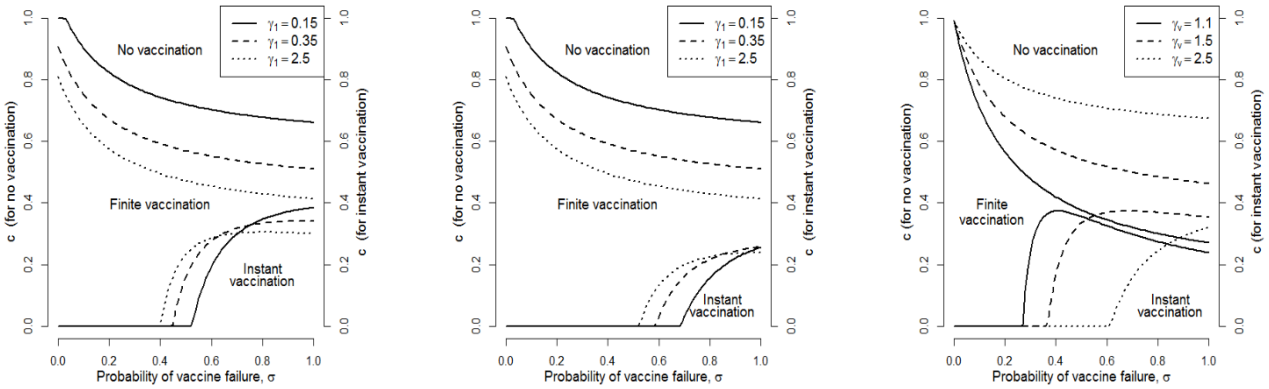
where: $b_2 = \gamma_1\gamma_v[(\sigma - 1)\gamma_u - \omega]$
$b_1 = (\sigma - 1)\gamma_u[(\sigma\gamma_u + \omega)\gamma_1 + k_6] + \omega(\gamma_1\gamma_u - k_6)$
$b_0 = \sigma\gamma_1\gamma_u - k_6$
$k_6 = \gamma_v(\gamma_1 + \omega)$
$k_7 = \sigma\gamma_1\gamma_u(\sigma - 1)\omega^3$

Theoretically, the fold bifurcation is expected to occur at the root(s) of the quadratic equation (15) which exists in an interval $0 \le c \le 1$.

III. Results and Discussion

A. The cost thresholds for no, finite and instant vaccination

Based on the Equation (11), with each corresponding pair of cost thresholds, we could divide the c-σ plane into three different regions, namely the regions for no vaccination, finite vaccination and instant vaccination, respectively. Whenever both c and σ values fall under the region of no (resp. instant) vaccination, we concluded that no one (resp. everyone) in the population will choose vaccination strategy, at (Nash) equilibrium.



(a) Various values of γ_1, and $\theta = 0.85$ (b) Various values of γ_1, and $\theta = 0.65$ (c) Various values of γ_v, and $\theta = 0.85$

Figure 1: The cost thresholds for no vaccination and instant vaccination.

Considering that the force of infection λ^* in the limit of $\bar{\pi} \to \infty$ will be non-negative whenever $\sigma\theta\beta\gamma_1 > \gamma_v(\gamma_1 + \omega)$ (see Equation (5), let $\omega \equiv \omega_V$), we should view the region of instant vaccination as an unfavourable phenomenon in the context of infectious diseases control. This is due to the fact that the region of instant vaccination corresponds to a circumstance whereby even though every susceptible in the population vaccinated instantly, the disease eradication would still not be achieved. That is, the effective reproduction number could not be further reduced to below its sub-threshold for disease eradication. The major factor contributing to this could be the vaccine efficacy. Unless otherwise specified, the parameter values $(\beta, \theta, \gamma_1, \gamma_u, \gamma_v, \omega)$ = (6, 0.85, 0.15, 1, 2, 0.05) are used for all numerical simulations in this paper.

Since Equation (12) is a decreasing function of σ, the impact of vaccine efficacy on the case of no vaccination could be easily understood. For a specific vaccine efficacy (i.e. $1 - \sigma$), the longer duration the vaccinated individuals acquire high immunity (i.e. smaller γ_1), the higher the cost threshold for no vaccination, c_{no} (three upper curves in Figure 1(a) and Figure 1(b)) will be. As far as the disease control is concerned, the higher c_{no}, the better it will be simply because individuals will not get vaccinated if the relative cost of vaccination, c, exceeds c_{no}. From an individual perspective, higher c_{no} implies that a player's utility is lower if he/she chooses not to vaccinate, and hence a rational individual is most likely to vaccinate when the rate of γ_1 is low. It is also worth noting that the Nash equilibrium vaccination rate is always finite when the parameter value γ_1 is small enough (say, $\gamma_1 = 0.15$) and the vaccine efficacy is almost perfect (i.e. $\sigma \to 0$).

By examining Figure 1(a) in which $\theta = 0.85$, we note that the instant vaccination will occur for $\sigma > 0.4$. As for intermediate values of σ, instant vaccination occurs for lower relative cost if the vaccinated individuals reside longer in the V_1 class. Ironically, whenever $\sigma \to 1$, the smaller γ_1, the higher relative cost of vaccination will be for instant vaccination threshold. This could be attributed to the mixing of vaccinated individuals with two classes of vaccine-induced immunity in the population. That is, the model (1) are more realistic by including assumptions that the vaccine would offer, on one hand, full protection to individuals with high immunity and on the other hand, partial protection to individuals with low immunity.

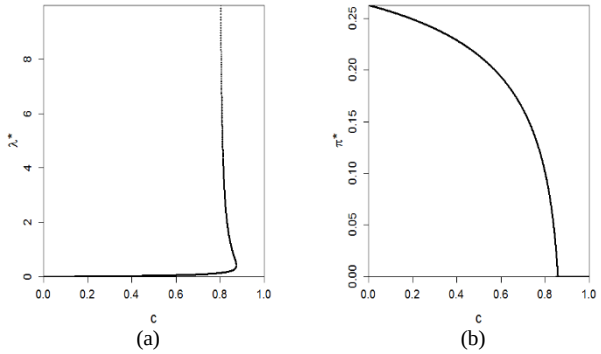
As the cost threshold for no vaccination (i.e. Equation (12)) is independent of θ, the three upper curves in Figure 1(a) and Figure 1(b) are identical. Since smaller θ, for instance θ = 0.65, implies that the greater reduction of transmissibility for vaccinated infected individuals, it follows that whenever the vaccinated individuals reside longer in the V_1 class, the instant vaccination occurs but the diseases may not be eradicated, for lower c and narrower range of low vaccine efficacy (Figure 1(b)). If vaccine could offer greater reduction of transmissibility to individuals in the V_1 class, then vaccines would bring greater benefit to the population than to the individuals who pay for the cost of vaccination. For that reason, opting for vaccination strategy voluntarily could be viewed as an altruistic behaviour [18], to some extent. If most susceptibles are altruistic, the spontaneous vaccination rate will eventually reach the social optimum [19]. In contrast with the assumption of selfishness in the classical game theory, indeed, it can be suggested that altruism play an important role in reducing the possibility of the coexistence of instant vaccination and disease prevalence with the reduction transmissibility, θ, as an additional characteristic of vaccine in our model even if the vaccine is not fully perfect.

Figure 1(c) illustrates the effect of γ_v on the Nash equilibrium vaccination rates. The most striking feature of this graph is that when γ_v increases (i.e. faster recovery for vaccinated infected individuals), there is a significant upward shift for the cost threshold of no vaccination, particularly for higher probability of vaccine failure. This reflects that if vaccine is able to protect vaccinated individuals in the V_2 class from being infected, people will probably show minimal concern on the duration of infection in choosing their vaccination strategy. On the other hand, when vaccine does not reduce the vaccinated individuals' susceptibility considerably, then the more the vaccine is able to speed up recovery in breakthrough infection, the higher the cost threshold for no vaccination will be. This implies that individuals will not refuse to vaccinate even though the relative cost of vaccination is high (i.e. low utility) whenever γ_v value is large. Also, we observe that the unfavourable instant vaccination occurs for lower c but larger σ when the duration of infection for individuals in the V_2 class is shortened.

B. Multiple Equilibria of Vaccination Rates

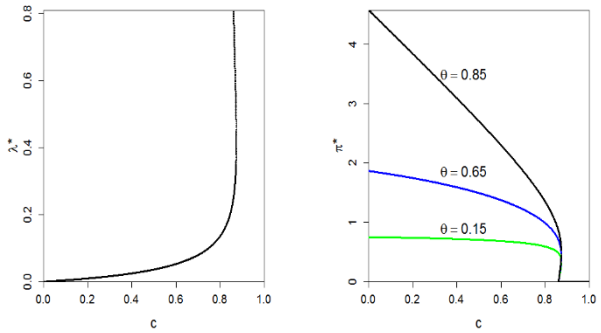
We solve the cubic Equation (10) numerically for $\sigma = 0.15$. Then, the graph of equilibrium force of infection λ^* against relative cost of vaccination, c, is plotted in Figure 2(a). There

exists at most two endemic equilibria. Besides that, the numerical simulation in Figure 2(a) and Equation (15) both give the fold bifurcation at c_{fold} = 0.874. When the c value is greater than c_{fold}, no feasible endemic equilibrium is found. However, when the c value is smaller than and close to c_{fold}, two endemic equilibria appear. By substituting λ^* obtained into the quadratic Equation (3), and solving for $\bar{\pi}$, the Nash equilibrium vaccination rate versus c is depicted in Figure 2(b). It can be seen that neither instant vaccination nor multiple Nash equilibria vaccination rates are found for the parameter values used in Figure 2 in our vaccination population game with two-class vaccine-induced immunity.



(a) (b)

Figure 2: Dependence of λ^* and π^* on c, for $\beta = 6$ and $\sigma = 0.15$



(a) (b)

Figure 3: Dependence of λ^* and π^* on c, for $\beta = 18$ and $\sigma = 0.15$

We then explore the extent to which a highly contagious disease could alter the individual vaccination decision-making by increasing the disease transmission rate from β = 6 to $\beta = 18$, for $\theta = \{0.85, 0.65, 0.15\}$. As $\lambda^*(\bar{\pi} = 0) = 0.810$, in Figure 3(a), we discard the graph whenever the numerical simulation produces $\lambda^* > 0.810$. Likewise, the λ^* in Figure 3(a) shows the fold bifurcation does occur. This complicates the individual vaccination decision-making in the cases of β = 18 whereby three Nash equilibria vaccination rates (i.e. two non-zero vaccination rates and one zero vaccination rate) appear in the interval $0.861 \leq c \leq 0.874$ (Figure 3(b)). Since θ is only implicitly appear in quadratic equation (3) through $\lambda^* = \beta \frac{I^* + \theta W^*}{N^*}$, Figure 3(b) reveals that the greater reduction of transmissibility for vaccinated infected individuals (i.e. smaller θ), the lower possibility that the unfavourable phenomenon of instant vaccination coexists with prevalent infectious diseases will be. For example, although its corresponding λ^* are non-zero when the c values are small for all three values of θ, we find that the Nash equilibrium strategy for $\theta = 0.15$ is finite vaccination (i.e. $\pi^* < 1$), whereas for $\theta \geq 0.65$, the instant vaccination (i.e. $\pi^* > 1$) occurs.

IV. Conclusion

By using a two-class vaccine-induced immunity SVIRS model in the framework of vaccination population games, we find that the additional characteristics of imperfect vaccine may alter the individuals' best response for vaccination strategy and consequently its epidemic dynamics. These characteristics complicate individuals vaccination behaviour and should not be overlooked in the effort of controlling infectious diseases by voluntary vaccination programme.

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