Analysis of Performance Measures with Single Channel Fuzzy Queues under Two Class by Ranking Method

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***Abstract*—In this paper, we propose a procedure to find different performance measures in terms of crisp value for new single fuzzy queues FM/F(H_1,H_2)/1 with two classes of arrival. The arrival and service rate are fuzzy numbers which represented by triangular, trapezoidal and hexagonal membership functions. The main idea is to obtain the exact crisp values from the fuzzy values, which is more realistic and practical in queueing system. The left and right ranking method is adopted to remove the fuzziness before the performance measures are computed by using conventional queueing theory. The main advantage of this approach, that it is simplicity in application and capability in discovering exact real data around fuzzy values. Numerical example is presented to illustrate the validity application of the procedure.**

***Index Terms*—Fuzzy Queues; Left and Right Ranking Method; Single Channel Priority; Two-Classes.**

I. INTRODUCTION

Queueing models have a wide application in service organizations and one of such application area is real life situations having a policy of two class service channels. Different types of queuing models have been explored in [1, 2] ranging from queuing models having constant crisp values to queuing models with uncertain or fuzzy values. However, practical problems have the assumptions that certain parameters in the queueing model are not constantly known as assumed by classical model. Therefore, Zadeh's principle [3] described certain models as possibilistic having expressions such as “the average arrival rate is approximately 10” or “service times are approximately 20”, which is a more realistic way to represent these values. This makes fuzzy queueing models more practical than the classical queueing models in many real situations.

On the other hand, the conversion of fuzzy queues to crisp queues has also been extensively discussed in literature and a number of methods and approaches have been put in use. One of such methods is the ranking method [4]. Another method that dates further back into literature is the robust ranking method [5], which was also studied by [6, 7]. The authors here adopted this method with single channel priority queuing models, while more recent studies [8-10] adopted priority queueing models with fuzzy queues. However, most previous studies have not considered two classes of arrival rates and two exponential service rates under arrival policy of first come first service (FCFS). Hence, this paper adopts one of the queueing models with the method for conversion from fuzzy to crisp values known as the left and right ranking method in application to three types of membership functions; triangular, trapezoidal and hexagonal membership functions. This leads to obtaining different performance measures in terms of crisp values for the fuzzy queuing model with two classes of arrival rates and mixture of exponential service rates.

The outline of this paper follows: Section 1 contains an introductory overview, Section 2 explains the basic concepts in fuzzy set theory, Section 3 describes the formulation of fuzzy queuing model with two classes and Section 4 introduces the left and right ranking method. Section 5 gives the numerical examples, the results and discussion while Section 6 concludes the article.

II. BASIC CONCEPTS OF FUZZY SET THEORY

A fuzzy set is specified by a membership function containing the components of a domain space or universe X in the interval [0, 1], that is $\widetilde{A} = \left\{\left(z, \mu_{\widetilde{A}}(z)\right); z \in Z\right\}$. Here, $\tilde{\mu}_{\widetilde{A}}: Z \to [0,1]$ is an interval called the degree of membership function of the fuzzy set $\widetilde{A}$ and $\mu_{\widetilde{A}}(z)$ represents the membership value of $z \in Z$ in the fuzzy set $\widetilde{A}$. Also, these membership degree are defined by $R \to [0,1]$.

A fuzzy set $\widetilde{A}$ of a universe of discourse X is called a normal fuzzy set if there exists at least z∈Z such that $\mu_{\tilde{A}}(z) = 1$.

A fuzzy set $\widetilde{A}$ is convex if and only if for any $z \in Z$ the membership function of $\tilde{A}$ satisfies the condition $\mu_{\tilde{A}}\{z_1 + (1-\lambda)z_1\} \geq min\{\mu_{\tilde{A}}(z_1), \mu_{\tilde{A}}(z_2)\}, 0 \leq \lambda \leq 1$.

A. Triangular Fuzzy Number

A triangular fuzzy number $\widetilde{A}(z)$ can be represented by $\widetilde{A} = (a, b, c; 1)$ where, a, b and c represent the points inside the interval with the membership function $\mu_{\widetilde{A}}(z)$ defined by:

$$\mu_{\widetilde{A}}(z) = \begin{cases} \dfrac{(z-a)}{(b-a)}, & a \leq z \leq b \\ 1, & z = b \\ \dfrac{(c-z)}{(b-c)}, & b \leq z \leq c \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

B. Trapezoidal Fuzzy Number

A trapezoidal fuzzy number $\tilde{A}(z)$ can be represented by $\tilde{A} = (a, b, c, d; 1)$ where, a, b, c and d represent the points inside the interval. The membership function $\mu_{\tilde{A}}(z)$ is defined by:

$$\mu_{\tilde{A}}(z) = \begin{cases} \frac{(z-a)}{(b-a)}, & a \le z \le b \\ 1, & b \le z \le c \\ \frac{(d-z)}{(d-c)}, & c \le z \le d \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

C. Hexagonal Fuzzy Number

A fuzzy number $\tilde{A}_H$ is an hexagonal fuzzy number denoted by $\tilde{A}_H = (a, b, c, d, e, f; 1)$, where a, b, c, d, e and f are real numbers [11], its continuous membership function $\mu_{\widetilde{A_H}}(z)$ is given by:

$$\mu_{\tilde{A}_H}(z) = \begin{cases} 0, & z < a \\ \frac{1}{2}\left(\frac{z-a}{b-a}\right), & a \le z \le b \\ \frac{1}{2} + \frac{1}{2}\left(\frac{x-b}{c-b}\right), & b \le z \le c \\ 1, & c \le z \le d \\ 1 - \frac{1}{2}\left(\frac{z-d}{e-d}\right), & d \le z \le e \\ \frac{1}{2}\left(\frac{f-z}{f-e}\right), & e \le z \le f \\ 0, & z > f \end{cases} \quad (3)$$

where, a, b, c, d, e and f represents the points inside the closed interval.

D. Generalized Fuzzy Number

Generalized fuzzy number is a fuzzy set of $\tilde{A}$, described in the universal set of real numbers represented R. Also, if the membership function characterized as $\mu_{\tilde{A}}: R \to [0, w]$ is continuous, $\mu_{\tilde{A}}(z) = 0$ for all $z \in (-\infty, a_1] \cup [a_4, \infty)$, then:

- $\mu_{\tilde{A}}(z)$ is strictly increasing on $[a_1, a_2]$; also strictly decreasing on $[a_3, a_4]$;
- $\mu_{\tilde{A}}(z) = w$ for all $z \in [a_2, a_3]$;
  where $0 < w \le 1$.

E. Left and Right Type Generalized Fuzzy Number

Generalized left and right fuzzy number is represented as $(a, b, c, d; w)_{LR}$. A fuzzy number can be described to be a left and right type generalized fuzzy number using many types of linear membership functions including:

a. Generalized Triangular Fuzzy Number

Generalized triangular fuzzy number $\tilde{A}(z)$ can be represented by $\tilde{A} = (a, b, c; w)$ under membership function $\mu_{\tilde{A}}(z)$ defined by:

$$\mu_{\tilde{A}}(z) = \begin{cases} \frac{w(z-a)}{(b-a)}, & a \le z \le b \\ w, & z = b \\ \frac{w(z-c)}{(b-c)}, & b \le z \le c \\ 0, & o.w \end{cases} \quad (4)$$

b. Generalized Trapezoidal Fuzzy Number

Generalized trapezoidal fuzzy number as $\tilde{A}(z)$ can be represented by $\tilde{A} = (a, b, c, d; w)$ under membership function $\mu_{\tilde{A}}(z)$ defined by:

$$\mu_{\tilde{A}}(z) = \begin{cases} \frac{w(z-a)}{(b-a)}, & a \le z \le b \\ w, & b \le z \le c \\ \frac{w(z-d)}{(d-c)}, & c \le z \le d \\ 0, & o.w \end{cases} \quad (5)$$

c. Generalized Hexagonal Fuzzy Number

Generalized hexagonal fuzzy number as $\tilde{A}(z)$ can be represented by $\tilde{A} = (a, b, c, d, e, f; w)$ under membership function $\mu_{\tilde{A}_H}(z)$ defined as:

$$\mu_{\tilde{A}_H}(z) = \begin{cases} 0, & z < a \\ \frac{1}{2}\left(\frac{w(z-d)}{(b-a)}\right), & a \le z \le b \\ \frac{1}{2} + \frac{1}{2}\left(\frac{w(z-b)}{(c-b)}\right), & b \le z \le c \\ w, & c \le z \le d \\ 1 - \frac{1}{2}\left(\frac{w(z-d)}{e-d}\right), & d \le z \le e \\ \frac{1}{2}\left(\frac{w(f-z)}{f-e}\right), & e \le z \le f \\ 0, & z > f \end{cases} \quad (6)$$

III. FUZZY QUEUING MODEL WITH TWO CLASSES

Consider a single channel fuzzy queueing model with two classes FM/F (H_1,H_2)/1/FCFS with no priorities in arrival rates, where FM denotes the fuzzy arrival rates as a Poisson process while F(H_1,H_2) denotes the fuzzified hyper exponential service time rates with two classes in a First Come First Serve (FCFS) manner, noting that the system capacity and population size is infinite.

In this model, customers arrived in groups by a single channel represented by $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\mu}_1$ and $\tilde{\mu}_2$ respectively. Let $\emptyset_{\tilde{\lambda}_1}(w), \emptyset_{\tilde{\lambda}_2}(x), \emptyset_{\tilde{\mu}_1}(y)$, and $\emptyset_{\tilde{\mu}_2}(z)$ then be the fuzzy sets represented by four sets as:

$$\tilde{\lambda}_1 = \left\{\left(w, \emptyset_{\tilde{\lambda}_1}(w)\right) \middle| w \in W\right\} \quad (7)$$

$$\tilde{\lambda}_2 = \left\{\left(x, \emptyset_{\tilde{\lambda}_2}(x)\right) \middle| x \in X\right\} \quad (8)$$

$$\tilde{\mu}_1 = \left\{\left(y, \emptyset_{\tilde{\mu}_1}(y)\right) \middle| y \in Y\right\} \quad (9)$$

$$\tilde{\mu}_2 = \left\{\left(z, \emptyset_{\tilde{\mu}_2}(z)\right) \middle| z \in Z\right\} \quad (10)$$

where *W*, *X*, *Y* and *Z* are crisp universal group of the arrival rate and service rate. Likewise, let *f(w,x,y,z)* denote the particular system of interest. Hence *w*, *x*, *y* and *z* are fuzzy numbers and likewise *f(w,x,y,z)* are fuzzy numbers.

Let $L_q^{(1)}$ and $L_q^{(2)}$ represent the conventional equation in the classical single queueing model as:

$$L_q^{(1)} = \frac{\lambda_1\left(\rho_1/\mu_1 + \rho_2/\mu_2\right)}{1-\rho}, \quad (11)$$

$$L_q^{(2)} = \frac{\lambda_2\left(\rho_1/\mu_1 + \rho_2/\mu_2\right)}{1-\rho}. \quad (12)$$

Also, the stability steady state is $\rho \equiv \rho_1 + \rho_2 < 1$ and $0 < \rho < 1$. The other performance measurements are defined by:

$$W_q^{(i)} = \frac{L_q^{(i)}}{\lambda_i} \quad (13)$$

$$W_s^{(i)} = W_q^{(i)} + \frac{1}{\mu_i} \quad (14)$$

$$L_s^{(i)} = \lambda_i W_s^{(i)}\ ;\ i = 1,2 \quad (15)$$

IV. LEFT AND RIGHT RANKING METHOD

In this section the steps to convert the fuzzy numbers into crisp numbers is explained. Three types of fuzzy numbers; triangular, trapezoidal and hexagonal fuzzy numbers are implemented with the left and right ranking procedure which is represented by $\mathrm{F(R)} \rightarrow \mathrm{R}$, $[12-13]$.

To start mapping the algorithm for this method, consider the triangular fuzzy numbers $\tilde{\mathrm{A}}(\mathrm{z}) = (\mathrm{a,b,c;w})$, then the left and right ranking is defined by:

$$R(\tilde{A}) = \int_0^w \frac{L^{-1}(z) + R^{-1}(z)}{2}, \quad (16)$$

where:

$$L^{-1}(z) = a + \left(\frac{b-a}{w}\right) \quad (17)$$

and:

$$R^{-1}(z) = b + \left(\frac{c-b}{w}\right) z, \quad (18)$$

$$R(\tilde{A}) = \frac{w(a+2b+c)}{4}. \quad (19)$$

In the same way, consider the trapezoidal fuzzy numbers:

$$\tilde{A}(z) = (a,b,c,d;w)$$
$$R(\tilde{A}) = \frac{w(a+b+c+d)}{4} \quad (20)$$

and the hexagonal fuzzy numbers $\tilde{A}(z) = (a,b,c,d,e,f;w)$ where:

$$\mathrm{L}^{-1}(\mathrm{z}) = \frac{1}{2}\left(\frac{\mathrm{w-a}}{\mathrm{b-a}}\right) + \frac{1}{2} + \frac{1}{2}\left(\frac{\mathrm{w-b}}{\mathrm{c-b}}\right) \quad (21)$$

$$\mathrm{R}^{-1}(\mathrm{z}) = 1 - \frac{1}{2}\left(\frac{\mathrm{w-d}}{\mathrm{e-d}}\right) + \frac{1}{2}\left(\frac{\mathrm{f-w}}{\mathrm{f-e}}\right) \quad (22)$$

By substituting in equation (16), the left and right ranking of hexagonal fuzzy numbers, the following is obtained:

$$R(\tilde{A}) = \frac{w(2a+3b+4c+4d+3e+2f)}{18} \quad (23)$$

V. RESULTS AND DISCUSSION

Consider a production line of machine receiving two types of arrival customers; $\tilde{\lambda}_1, \tilde{\lambda}_2$ and the service time represented as a mixture of the exponential distribution $\tilde{\mu}_1$, $\tilde{\mu}_2$ respectively. Note that all parameters are in a fuzzy environment and the management wants to compute the mean of queue length for each class. The three types of fuzzy numbers considered are triangular, trapezoidal and hexagonal fuzzy numbers as illustrated for the method in the following subsections.

A. The Triangular Membership Function

Assume that both arrival rate with two classes and service rates are given as $\tilde{\lambda}_1 = [20,30,40]$, $\tilde{\lambda}_2 = [60,70,80]$, $\tilde{\mu}_1 = [110,120,130]$ and $\tilde{\mu}_2 = [150,160,170]$. Through equation (19) the ranking of class one and class two are given as:

$$R(\tilde{\lambda}_2) = R(20,30,40;1) = \frac{(20+60+40)}{4} = 30 \quad (24)$$

$$R(\tilde{\lambda}_2) = R(60,70,80;1) = \frac{(60+140+80)}{4} = 70 \quad (25)$$

$$R(\tilde{\mu}_1) = R(110,120,130;1) = \frac{(110+240+130)}{4} = 120 \quad (26)$$

$$R(\tilde{\mu}_2) = R(150,160,170;1) = \frac{(150+320+170)}{4} = 160 \quad (27)$$

According to Equations (11) and (12):

$$L_q^{(1)} = \frac{30\left(\frac{30}{120}/120 + \frac{70}{160}/160\right)}{1 - \frac{30+70}{120+160}} = 0.225 \quad (28)$$

$$L_q^{(2)} = \frac{70\left(\frac{30}{120}/120 + \frac{70}{160}/160\right)}{1 - \frac{30+70}{120+160}} = 0.524 \quad (29)$$

B. The Trapezoidal Membership Function

Assume that both the arrival rate with two class and service rates are trapezoidal fuzzy numbers defined as:

$\tilde{\lambda}_1 = [20,30,40,50]$
$\tilde{\lambda}_2 = [60,70,80,90]$
$\tilde{\mu}_1 = [110,120,130,140]$
$\tilde{\mu}_2 = [150,160,170,180]$

According to Equation (20) the ranking of class one and class two are given as:

$R(\tilde{\lambda}_1) = 35$
$R(\tilde{\lambda}_2) = 75$
$R(\tilde{\mu}_1) = 125$
$(\tilde{\mu}_1) = 165$

C. The Hexagonal Membership Function

Assume that both arrival rate with two classes and service rates are hexagonal fuzzy numbers are given as:

$\tilde{\lambda}_1 = [20,30,40,50,60,70]$
$\tilde{\lambda}_2 = [60,70,80,90,100,110]$
$\tilde{\mu}_1 = [110,120,130,140,150,160]$
$\tilde{\mu}_2 = [150,160,170,180,190,200]$

Through Equation (23) the ranking of class one and class two are given as:

$R(\tilde{\lambda}_1) = 45$
$R(\tilde{\lambda}_2) = 85$
$R(\tilde{\mu}_1) = 135$
$R(\tilde{\mu}_2) = 175$

We obtain $L_q^{(1)} and\, L_q^{(2)}$ for trapezoidal and hexagonal fuzzy numbers in similar manner. Recall Equations (13), (14) and (15) to evaluate the whole system via other performance measurements. The results obtained are given in Table 1 which explains different values of each class for all types of membership functions considered (triangular, trapezoidal and hexagonal fuzzy numbers) as:

Table 1
Different Performance Measurements of Three Types of Membership Functions

MFT	$L_q^{(1)}$	$L_q^{(2)}$	$W_q^{(1)}$	$W_q^{(2)}$	$W_s^{(1)}$	$W_s^{(2)}$	$L_s^{(1)}$	$L_s^{(2)}$
I	0.225	0.524	0.007	0.008	0.015	0.031	0.47	0.91
II	0.282	0.604	0.006	0.007	0.014	0.016	0.56	0.75
III	0.196	0.366	0.003	0.004	0.010	0.011	0.52	0.85

where:
MFT denotes Membership Function Type
I denotes Triangular Membership Function
II denotes Trapezoidal Membership Function
III denotes Hexagonal Membership Function

From the results displayed in Table 1, it is clear that the ranking method gives various sets of real values such as arrival rates and service rates for each class. Likewise, different performance measurements are obtained which are given and is seen to converge between two classes in the whole system. It is also seen from Table 1 that all performance measurements of class one are less than performance measures of class two in the system for both three types of fuzzy numbers (triangular, trapezoidal and hexagonal). Also, the using of more types of fuzzy numbers leads us to obtaining more real data and flexible choices in the system.

VI. Conclusion

In this paper, fuzzy set theory is shown to be a strong tool when dealing with real applications in queuing models with two classes such as manufacturing production line. The ranking approach adopted is also seen to be effective when transforming fuzzy queues into crisp queues, thus evaluating the system by conventional performance measurements such as the expected queue length of customers in the queues and system for both classes of arrivals. Also, the computation of the expected waiting time of customers in queue and in the whole system is obtained too. Therefore the manager can take the best values and make optimal decisions. Another advantage of using ranking method index is obtaining exact values inside closed crisp interval, while also providing more than one solution of values in the queueing system with different types of membership functions. For future work in this area, researchers can investigate the effectiveness of this approach to other queueing models and other types of linear membership functions.

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