Intersection of Main James Abacus Diagram for the Outer Chain Movement with Length [1, 0, 0…]

Eman F. Mohommed[1], Nazihah Ahmad[1], Haslinda Ibrahim[2]
[1]*School of Quantitative Sciences, College of Arts & Sciences, Universiti Utara Malaysia, 06010 Sintok, Kedah, Malaysia.*
[2]*Department of Mathematics College of Science University of Bahrain P.O. Box 32038 Kingdom of Bahrain.*
manfatel@yahoo.com

***Abstract*—James abacus with β- number is one of the graphical representations for any partition of a non-negative integer. James abacus can be divided into several chains which consist of outer and inner chains. In this paper, a new diagram A^{tco} is developed by employing movement to the outer chain with length [1, 0, 0,…] on the active James abacus. Then, a consecutive new diagram of b_1, b_2…, b_{e-1} can be found from active diagram A^{tco} . Finally intersection of the main new diagrams A^{tco} is obtained. Some theoretical results for intersection of main James diagram are established. In addition, several examples are given to illustrate the results.**

***Index Terms*—James Abacus; Partition; Beta Number.**

I. Introduction

James abacus diagram is an important component in the modern algebra, and plays a key role in Iwahori−Hecke and *q*-Schur algebras [1], [2], [3], [5], [11], [13]. James abacus diagram is the quickest way to represent partitions of a non-negative integer. This diagram has been introduced by G. D. James in 1978. It is a new version of Young diagram with a new addition condition for the number of runners (columns), e where $e \geq 2$. James proposed his diagram by employing rim hook idea to the young diagram [3]. A set of β- number for a diagram where $\beta = (\mu_1+b-1,\ \mu_2+b-2,\dots,\ \mu_j+b-j)$ is added to represent the partition $\mu=(\mu_1, \mu_2, \dots, \mu_n)$ where has some zero parts at the end of partition. The numbers of β-number depended on number of the original partition parts, b. The numbers b must be greater than or equal to the number of parts of original partition. This β-number can be rearranged according to e. Thus, every partition can be represented by infinite number of James abacus. Several diagrams have been constructed depending on James abacus such as e+1-abacus, e-1-abacus, k - abacus, diagram W^*, single diagram, diagram A', diagram A'', diagram A^1 diagram A^2, diagram A^3, diagram A^4, diagram A^5, diagram A^6, diagram A^7. A^{tc} Those diagram can be applied by adding one empty runner [4], adding full runner [1], removing one runner [2], scan movement [14], single-step movement, upside down transformation [8], right-side- left transformation, direct rotation transformation, upside down direct rotation. Transformation, right side-left direct rotation transformation [9, 10]. We has developed chain movement for e = 2 in [12]. In this paper, a new diagram $\boldsymbol{A^{tco}}$ is developed by employing movement to the outer chain with length [1, 0, 0,…] on the active James abacus for any e . Location of all the beads in the Active James abacus will be changed accordingly when the initial beta numbers are moved. Diagram A^{tco} be denoted to new. Finally, intersection of the main diagrams is constructed.

II. Prelimiaries

Let $\mu = (\mu_1, \mu_2, \dots, \mu_j)$ is a partition of positive integer number, such that $|\mu| = \sum_{i=1}^{j} \mu_i,\ \forall i \geq 1$ and choose an integer j such that $\mu_{j+1} = 0$. β-numbers were defined by fix μ is a partition of d, choose an integer b greater than or equal to the number of parts of μ and define $\beta_i = \mu_i + b - i$, $1 \leq i \leq b$. The set $\{\beta_1, \beta_1, \dots, \beta_b\}$ is said to be the set of β-numbers for μ [6].

Example 1. Let (6, 5, 4, 4, 3, 1), (5, 5, 5, 3, 2, 3), (7, 7, 6, 1, 1, 1) are a partitions of 23. Then β-numbers of partition (6, 5, 5, 4, 4, 3, 1) are {1, 4, 6, 7, 9, 10, 12}.

James abacus as shown in Figure 1 consist of an e positive integer number greater than or equal to 2, e vertical runner numbered 0,….,e-1 from left to right. On runner n we mark the position labeled n, $e+n$, $2e+n$,..., where $0 \leq n \leq e-1$ from the top to down. Thus, we can represent β-numbers with many runners depending on e.

run.0	run.1	run.2		run.e-1
0	1	2	……	e-1
e	e+1	e+2	……	2e-1
2e	2e+1	2e+2	……	3e-1
.	.		.	.
.	.		.	.
.	.		.	.

Figure 1: James abacus

Every β-number of James abacus also can be represented by a bead (o) position which takes its location in the diagram and (-) to represent empty bead position. We also enumerate the beads, so that bead 1 occupies by a occupies position $\mu_1 - 1 + b$, bead 2 a occupies position $\mu_2 - 2 + b$ so on. Recall example 1 where (6, 5, 5, 4, 4, 3, 1), a bead and empty position of James abacus are shown in Figure 1 and 2.

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19

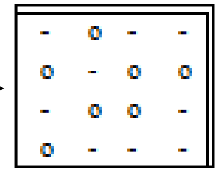


Figure 2: The positions on the abacus with 4 runners, the arrangement of beads (numbered) representing μ= (6, 5, 5, 4, 4, 3, 1).

A. *Active James Abacus*

In a partition one or more zero can be added to a partition such as $\mu^1 = (\mu_1, \mu_2, \dots, 0)$, $\mu^2 = (\mu_1, \mu_2, \dots, 0, 0)$, … , $\mu^f = (\mu_1, \mu_2, \dots, 0, \dots, 0)$ the sum elements consequently, the active James abacus is representing the original partition. If zero is added to the original partition, then it is inactive James abacus and note that β- number will also change.

Definition 1. A diagram that represents the original partition for any e greater than or equal to two is called active James abacus.

An example of active and inactive is given example 2.

Example 2 For above example when μ= (6, 5, 5, 4, 4, 3,1) active partition is μ^1= (6, 5, 5, 4, 4, 3, 1, 0), partition of b_2 is μ^2= (6, 5, 5, 4, 4, 3, 1, 0, 0) are given in Table 1.

Table 1
β- number where μ = (6, 5, 5, 4, 4, 3, 1), μ^1 = (6, 5, 5, 4, 4, 3, 1, 0), μ^2 = (6, 5, 5, 4, 4, 3, 1, 0, 0).

β_i	Active partition	partition of b_1	partition of b_2
β_1	12	13	14
β_2	10	11	12
β_3	9	10	11
β_4	7	8	9
β_5	6	7	8
β_6	4	5	6
β_7	1	2	3
β_8		0	1
β_9			0

Lemma 1. Let r be the number of active James abacus rows then:

$$r = \begin{cases} \dfrac{\beta_1}{e} & \text{if} \quad \beta_1 \text{ divide } e \\ \left\lfloor \dfrac{\beta_1}{e} \right\rfloor & \text{if} \quad \beta_1 \text{ does not divide } e \end{cases}$$

Proof:

By definition of active James diagram β_1 located in last rows and every rows have e positions then the number of active James abacus is $\frac{\beta_1}{e}$. If $\frac{\beta_1}{e}$ is decimal number, then we take the greatest integer**.** So the number of the active James abacus rows is $\left\lfloor \frac{\beta_1}{e} \right\rfloor$.

Remark 1. We combine repeated entries and use exponents to partition positive integer numbers. So, if τ be the number of redundant parts of the partition μ of t, then we have $(\mu_1, \mu_2, \dots, \mu_j) = (\lambda_1^{\tau_1}, \lambda_2^{\tau_2}, \dots, \lambda_m^{\tau_m})$ such that $\sum_{i=1}^{n} \mu_i = \sum_{j=1}^{m} \lambda_j^{\tau_j}$. Thus (5, 3, 3, 2, 1) can be rewrite as (5, 3^2, 2, 1).

B. *Main Diagram*

Every e have special partition { μ, μ^1 , μ^2 ,…, μ^{e-1} } called main partition. The diagram which is representing by main partition called main diagrams [7]. As can be seen on the table 1, the active James abacus is representing the original partition. If zero is added to the original partition, then it is inactive James abacus. Active James abacus play main role to find all James abacus of b_f. James abacus of b_1 can found direct from active James abacus, all the beads in each runner of the active James abacus will be transferred by right-shifted motion and in the same location to the James abacus of active James abacus except the beads within runner e-1, and they transfer to the first runner of the diagram that follows by down-shifted and added one bead up as the same step, b_2 can be form by b_1. From the above relationship, it can be concluded that the James diagram by b_1, b_2, · · ·, can be found from the active James diagram. Thus active James abacus with James abacus of b_f where f =1, 2, …, (e-1) called main diagrams [8].

In order to obtain a better understanding of main James abacus, definition 2 is needed.

Definition 2. Let $(me+n)_{aj}$ Is an active James abacus position, we define function $f\colon \mathbb{R} \to \mathbb{R}$ such that:

$$f(me+n)_{aj} = \begin{cases} (me+n+1)_{b_1} & \text{if} \quad 0 \le m \le r-1, 0 \le n \le e-1 \\ (m+1+0)_{b_1} & \text{if} \quad 0 \le m \le r-1, n = e-1, \end{cases}$$

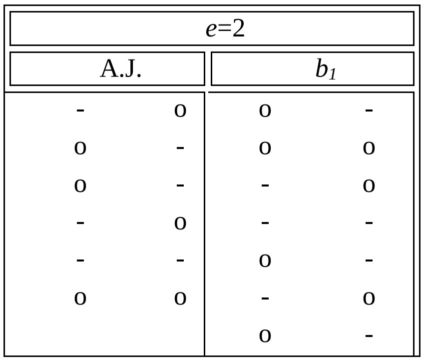
$(0e+0)_{b_1}$ is bead position, where $f(me+n)_{aj}$ is a position in James abacus of b_1.

Note that using Definition 2, be can obtained James abacus of b_1 from active James abacus and this also can be applied for the rest of b_{f+1} from b_f.

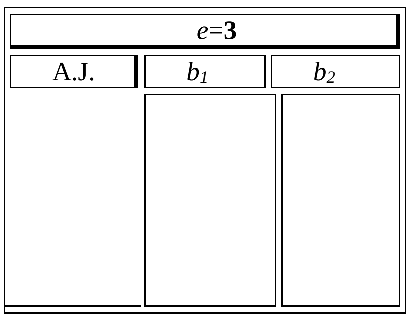
$$f(me+n)_{b_f} = \begin{cases} (me+n+1)_{b_{f+1}} & \text{if} \quad 0 \le x \le r-1, 0 \le y \le e-1 \\ (m+1+0)_{b_{f+1}} & \text{if} \quad 0 \le x \le r-2, y = e-1, \end{cases}$$

$(0e+0)_{b_f}$ is bead position

Example 3. If e=2 then μ= (6, 6, 4, 2, 1, 1) have two main diagrams, active James abacus and James abacus of b_1. While if e=3 then μ= (6, 6, 4, 2, 1, 1) have three main diagrams, active James abacus, James abacus of b_1 and James abacus of b_2. As shown in Figure 3.



(a)



(b)

Figure 1: Main diagram when μ = (6, 6, 4, 2, 1, 1) a) if e = 2 b) if e = 3

While the remaining diagrams are a repetition of one of the main diagrams. The remaining James abacus of b_{s+ek} can be easily obtained from the main James abacus of b_s and James abacus of b_{ek} can be obtained from active James abacus from, by adding k full rows above the b_s and Down-shifted, where s = 0,1,…,e and k integers number, see figure 4.

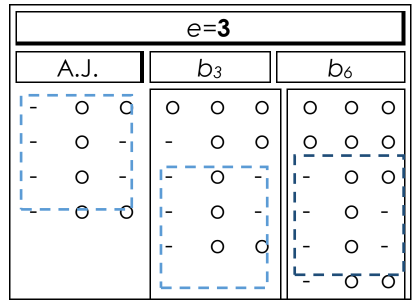
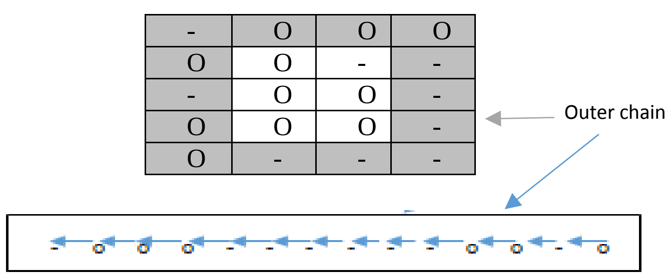


Figure 4: Illustration of down shift when e=3

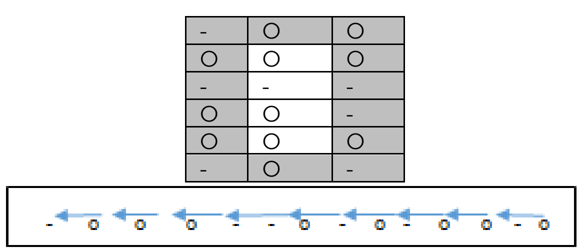
III. Outer Chain Movement

James abacus will be divided in to nested chain which consists of outer and inner chains. The main goal of this work is construct new diagram by application chain movement on the outer chain W ={n, re+n, me, (m+1)e-1: $0 \le n \le e$-1, $1 \le m \le r$-1, where $r = \left\lceil \frac{\beta_1}{e} \right\rceil$ and rest abacus positions called inner chains of the James abacus. Inner position will remain in the same position after applying chain movement on outer chain. To apply our movement**,** the positions will be rearranged as { 0,1,2,…, e-1, 2e-1, 3e-1,…, re-1, re-2, re-3,..., re-e, (r-1)e, (r-1)e+1, (r-3)e+2,…,re-1}. Each position moves one position exactly one step to the left except position '0' become position 'e-1' in new diagram.

Example 4. Let μ = (6, 5^3, 4^2, 1^5) is a partition of 34 and if e = 4 then outer chain which includes all bead and empty bead positions in the set W={0, 1, 2, 3, 7, 11, 15, 19, 18, 17, 6, 12, 8, 4}= {-, o, o, o, -, -, -, -, -, o, o, -, o}. While if e = 3 W= {0, 1, 2, 5, 8, 11, 14, 17, 16, 15, 12, 9, 6, 3} = {-, o, o, o, -, -, o, -, o, -, o, o, -, o}. Figure 5.a, 5.b represented to the position motion.



(a)



(b)

Figure 5: The position motion in outer chain for μ = (6, 5^3, 4^2, 1^5).when a) e=4 and b) e=3

Rule 1. Let r be the number of diagram rows for any partition μ of a non-negative integer t, e is great than or equal to 2, $(me+n)_{aj}$ any position in active James abacus, located in column n and rows m then:

$$(me+n)_{aj} \rightarrow \begin{cases} ((m+1)e+n)_{A^{tco}} & \text{if} \quad n=1, 0 \le \mathrm{m} \le \mathrm{r}-1 \\ (me+n+1)_{A^{tco}} & \text{if} \quad m=r-1, 0<\mathrm{n}<\mathrm{e}-1 \\ (me-1)_{A^{tco}} & \text{if} \quad n=e-1, 0 \le \mathrm{m} \le \mathrm{r}-1 \\ (me+n-1)_{A^{tco}} & \text{if} \quad m=0, 0 \le n < r-1 \end{cases}$$

Where $(me+n)_{A^{tco}}$ is an active diagram A^{tco} . $0 \le m \le, 0 \le r-1, n \le e-1$ and $r = \left\lceil \frac{\beta_1}{e} \right\rceil$. Figure 6 presented the rule 1 when μ= (5, 5, 4, 3, 2, 1) and e = 4.

Active diagram A^{tc}			
o	-	o	o
-	o	-	-
-	-	o	o

(a)

Active James			
-	o	-	o
-	o	-	o
-	o	o	-

(b)

Figure 6: a) Active James abacus b) Active diagram A^{toc} when μ= (5, 5, 4, 3, 2, 1) and e = 4.

Rule 2: Let β_i is a beta number of μ then:

Example 5: Let μ= (7, 6, 5, 3, 3 1, 1, 1), β-number = {1, 2, 3, 6, 7, 10, 13, 15}. By above rule β-number set become {0, 1, 2, 3, 6, 10, 11, 14}.

The active diagram A^{tco}, play a main role to design all the main diagrams A^{tco} in b_1, b_2,…, as following role 3.

Role 3: Let r be the number of diagram rows for any partition μ of a non-negative integer t, e is great than or equal to 2, $(me+n)_{aj}$ is any position in active James abacus, located in column n and rows m then:

$$\beta i \rightarrow \begin{cases} \beta_i - 1 & \text{if} \quad 0 < \beta i < e \\ \beta_i + 1 & \text{if} \quad (r-1)e \le \beta i \le e \\ \beta i + e & \text{if} \quad \beta i - e \\ \beta i - e & \text{if} \quad \beta i = ne - 1 \end{cases}$$

$$(me+n)_{aj} \rightarrow$$

$$\begin{cases} (me+n+1)_{b_1} & \text{if} \quad 0 \le n < e-2, m=0 \\ (2e)_{b_1} & \text{if} \quad m=0, n=e-2, 2 \le r \\ (rm+(3-r)n)_{b_1} & \text{if} \quad m=0, n=e-2, 2>r \\ ((m+2)e)_{b_1} & \text{if} \quad 0 \le m < r-1, n=e-1, \\ & \qquad m+2 \le r \\ (rm+(m+3-r)n)_{b_1} & \text{if} \quad 0 \le m < r-1, n=e-1, \\ & \qquad m+2>r \\ ((m-1)e+n)_{b_1} & \text{if} \quad m=r-1, \ge n=e-1 \\ (me+n)_{b_1} & \text{if} \quad m=r-1, 0<n<e-1 \\ ((m-1)e+n+1)_{b_1} & \text{if} \quad 1<m \le r-1, n=0 \\ 0 & \text{if} \quad m=1, n=0 \\ (me+n+1)_{b_1} & \text{if} \quad 1 \le n < e-2, 1 \le m < r-1 \\ (me-1)_{b_1} & \text{if} \quad n=e-2, 1 \le m < r-1 \end{cases}$$

Note that using rule 2 and 3, be can obtain b_1 from active diagram A^{tco} and this also can be applied for the rest of b_{f+1} from b_f, where f =1, 2,…, e-1.

We can explain this rule in Figure 7, where μ= (5, 5, 4, 3, 2, 1) and e = 4.

Active James	b_1	b_2	b_3
- o - o	o - o -	o o - o	o o o -
- o - o	o - o -	- o - o	o - o -
- o o -	o - o o	- o - o	o - o -
	- - - -	o - - -	o o - -
		- - - -	- - - -
			- - - -

(a)

Active diagram A^{tcc}	b_1	b_2	b_3
o - o o	- o - -	o - o o	o o - -
- o - -	o - o o	o o - o	o - o -
- - o o	o - o -	- o - -	o - o -
	o - - -	- - - -	o o - -
		o - - -	o - - -
			- - - -

(b)

Figure 7 (a) Active James (b) Active diagram A^{tcc} If the length of the diagram is [1,0] where μ= (5, 5, 4, 3, 2, 1) and e =4

IV. THE INTERESECTION OF MAIN DIAGRAMS

The idea of intersection given by [8] depending on its active James abacus, James abacus of b_1, James abacus of b_2, ..., James abacus of $b_{e\text{-}1}$. Later, this intersection idea is used in [6, 7]. In this paper, intersection of the main diagrams is constructed by considering the following two cases:

Case 1 : The old main partition diagrams.
Case 2 : The new main partition diagrams, i.e., after applying chain movement.

Case 1 : The old main partition diagrams.

Remark 2: The idea of intersection any main diagram is defined by the following:

1. We denote the intersection of main diagrams by $\cap_{s=1}^{e} m.d._{b_s}$.
2. The intersection's result as a numerical value will be denoted by $\#\cap_{s=1}^{e} m.d._{b_s}$, and it is equal to ϕ in the case of non-existence of any bead, or γ in the case that γ common beads exist in the main diagrams.

Theorem 1: [7] For any $e \ge 2$, the following holds:

1. $\#\cap_{s=1}^{e} m.d._{b_s}$=1 = ϕ if $\tau_k = 1, \forall\, k$ where $1 \le k \le m$.
2. Let Ω be the number of parts of λ which satisfies the condition $\tau_k \ge e$ for some k, then: $\#\cap_{s=1}^{e} m.d._{b_s} = [\sum_{t=1}^{\Omega} \tau_t - \Omega\,(e-1)]$.

For our example the intersection point of original main diagrams is $\#\cap_{s=1}^{e} m.d._{b_s} = [\sum_{t=1}^{\Omega} \tau_t - \Omega\,(e-1)]$= [2-1]=2.

Example 6:

1. Let μ=(6, 5, 2, 1) be a partition of 14, since τ_k = 1, we observe the non-Existence of any common beads. This led us to the following result: $\#\cap_{s=1}^{e} m.d._{b_s}$=0

2. Let μ= (5, 5, 4, 3, 2, 1). Using the relation mentioned in point 2 of the theorem, we will have the following:
   - If e = 2 $\#\cap_{s=1}^{e} m.d._{b_s}$ = [2-1]=1
   - If e=3 $\#\cap_{s=1}^{e} m.d._{b_s}$ = 0

Case 2 : The new main partition diagrams, i.e., after applying chain movement.

In order to obtain intersection of main diagram A^{tco} which stated in Theorem 3, Remark 3 are needed. For each cases in Remark 3 will have μ' and μ''.

Remark 3:

1. If $\tau_1 + B \geq e$ then $\mu' = (\mu_1^{\tau_1'}, \mu_2^{\tau_2'}, \dots, \mu_f^{\tau_f'})$ where $\tau_1' = B + \tau_1 - e, \tau_2' = \tau_2, \tau_3' = \tau_3, \dots, \tau_f' = \tau_f$. $\mu'' = (\mu_1^{\tau_1''})$ where $\tau_1'' = \tau_1 - \tau_1'$.
2. If $\tau_1 + B < e$ then choose the smallest v which achieve$[B + \sum_{\psi=1}^{v} \tau_\psi + (\mu_\psi - \mu_{(\psi+1)})] \geq e$.

If $[B + \sum_{\psi=1}^{v} \tau_\psi + (\mu_\psi - \mu_{(\psi+1)})] = e$,

then $\mu' = \left(\mu_{(v+!)}^{\tau'_{(v+1)}}, \mu_{(v+2)}^{\tau'_{(v+2)}}, \dots, \mu_f^{\tau_f'}\right), \mu'' = \left(\mu_v^{\tau_v''}, \mu_{(v-1)}^{\tau''_{(v-1)}}, \dots, \mu_1^{\tau_1''}\right)$

where: $\tau'_{(v+1)} = \tau_{(v+1)}, \tau'_{(v+2)} = \tau_{(v+2)}, \dots, \tau_f' = \tau_f$

$\tau_v'' = \tau_{v,} \tau''_{(v-1)} = \tau_{(v-1)}, \dots, \tau_1'' = \tau_1$.

If $[B + \sum_{\psi=1}^{v} \tau_\psi + \sum_{\psi}^{v}(\mu_\psi - \mu_{\psi+1})] > e$,

then $\mu' = (\mu_{(v+1)}^{\tau'_{(v+1)}}, \mu_{(v+2)}^{\tau'_{(v+2)}}, \dots, \mu_f^{\tau_f'}), \mu'' = (\mu_v^{\tau_v''}, \mu_{v-1}^{\tau''_{v-1}}, \dots, \mu_1^{\tau_1''})$

where $\tau'_{(v+1)} = \tau_{(v+1)}, \tau'_{(v+2)} = \tau_{(v+2)}, \dots, \tau_f' = \tau_f$

$\tau_v'' = \tau_{v,} \tau''_{(v-1)} = \tau_{(v-1)}, \dots, \tau_1'' = \tau_1$

If $[B + \sum_{\psi=1}^{v} \tau_\psi + \sum_{\psi}^{v-1}(\mu_\psi - \mu_{\psi+1}) > e$,

then $\mu' = (\mu_v^{\tau_v'}, \mu_{v+1}^{\tau'_{(v+1)}}, \dots, \mu_f^{\tau_f'}), \mu'' = (\mu_{v-1}^{\tau''_{v-1}}, \mu_{v-2}^{\tau''_{v-2}}, \dots, \mu_1^{\tau_1''})$

where $\tau_v' = \tau_v - e + \sum_{\psi=1}^{v-1} \tau_\psi + \sum_{\psi}^{v-1}(\mu_\psi - \mu_{\psi+1})$

$\tau'_{v+1} = \tau_{v+1}, \dots, \tau_f' = \tau_f$

$\tau_v'' = \tau_v - \tau_v', \tau''_{(v-1)} = \tau_{(v-1)}, \dots, \tau_1'' = \tau_1$.

$B = er - 1 - \beta_1$

Theorem 2: For any partition:

1. $\# \cap_{s=1}^{e} m.d._{b_s}$ μ' main diagramsA^{tco}= [$\sum_{\rho=1}^{\Omega} \tau_\rho' - \Omega\,(e - 1)$]
2. If $\tau_1 + B \geq e$ then $\# \cap_{s=1}^{e} m.d._{b_s}$ μ'' main diagrams A^{tco} = $\left[\sum_{k=1}^{\xi} \tau_k - e + 1\right] + \delta$.

Proof:

1. Since the common bead positions position of the James diagram in μ' will be stile in same location thus $\# \cap_{s=1}^{e} m.d._{b_s}$ μ' main diagrams A^{tco} equal to [$\sum_{\rho=1}^{\Omega} \tau_\rho' - \Omega\,(e - 1)$]
2. (a) If $\left[B + \sum_{\psi=1}^{v} \tau_\psi + (\mu_\psi - \mu_{(\psi+1)})\right] = e$
   then $\# \cap_{s=1}^{e} m.d._{b_s}$ μ''
   main diagramsA^{tco} = $\left[\sum_{k=1}^{\xi} \tau_k - e + 1\right] + \delta$.

   (b) $[B + \sum_{\psi=1}^{v} \tau_\psi + \sum_{\psi}^{v}(\mu_\psi - \mu_{\psi+1})] > e$
   then $\# \cap_{s=1}^{e} m.d._{b_s}$ μ''
   main diagramsA^{tco} = $\left[\sum_{k=1}^{\xi} \tau_k - e + 1\right] + \delta$.

   (c) If $[B + \sum_{\psi=1}^{v} \tau_\psi + \sum_{\psi}^{v-1}(\mu_\psi - \mu_{\psi+1}) > e$
   and $\tau_v \geq e - 1 + \tau_v''$
   then $\# \cap_{s=1}^{e} m.d._{b_s}$ μ''
   main diagrams A^{tco} = $\left[\sum_{k=1}^{\xi} \tau_k - e + 1\right] + \delta + \tau_v''$ -1.

   (d) If $[B + \sum_{\psi=1}^{v} \tau_\psi + \sum_{\psi}^{v-1}(\mu_\psi - \mu_{\psi+1}) > e$
   and $\tau_v < e - 1 + \tau_v''$
   then $\# \cap_{s=1}^{e} m.d._{b_s}$ μ''
   main diagramsA^{tco} = $\left[\sum_{k=1}^{\xi} \tau_k - e + 1\right] + \delta$.

where Ω be the number of parts of μ' which satisfies the condition $\tau_\rho' \geq e$ for some. δ be the number of parts of μ'' which satisfies the condition $\tau_\rho'' \geq e$ for some δ be the number of parts of μ'' which satisfies the condition $\tau_\rho'' = e - 1$ for some k.

Special case: If $\tau_v \geq e - 1 + \tau_v''$ and (r-2)m is bead position then $\# \cap_{s=1}^{e} m.d._{b_s}$ μ'' main diagramsA^{tco} = $\left[\sum_{k=1}^{\xi} \tau_k - e + 1\right] + \delta + \tau_v''$.

Example 7: Let μ = (9^3, 7^3, 3^3, 1) and e=3

To find the common bead positions or the intersection point of main diagram easily we will transfer James abacus of several rows into one row based on the idea One-Runner abacus. We can also found the number of the bead position by used last theorem

$\# \cap_{s=1}^{e} m.d._{b_s}$ μ' main diagramsA^{tc}= [$\sum_{\rho=1}^{\Omega} \tau_\rho' - \Omega\,(e - 1)$] + $\# \cap_{s=1}^{e} m.d._{b_s}$ μ''=2+1=3

Remark 4: The numerical value of intersection point depended on the ***e*.** The relationship between numerical value and e is Inverse relationship. We will depicted this relationship in Figure 7 and Table 2 respectively, where μ = (5, 3^2, 2, 1), e=3 and e=4. Notice that $\# \cap_{s=1} m.d._{b_s}$=1 and $\# \cap_{s=1}^{2} m.d._{b_s}$=0.
Now after application outer chain movement will be get a new partition. To find this new partition

Identity 1: For $1 \leq i < b$

1. $\mu_b^{A^{tco}} = \beta_b^{A^{tco}}$.
2. $\mu_i^{A^{tco}} = \mu_{(i+1)}^{A^{tco}} + \beta_i^{A^{tco}} - \beta_{i+1}^{A^{tco}} - 1$.

Example 8: Recall to the example 2, μ= (5, 5, 4, 3, 2, 1), β-number = {10, 9, 7, 5, 3, 1}.

By rule 2 $\beta^{A^{tco}} = \{11, 10, 5, 3, 2, 0\}$. By Identity 1

1. $\mu_6^{A^{tco}} = \beta_6^{A^{tco}} = 0$.
2. $\mu_5^{A^{tco}} = \mu_6^{A^{tco}} + \beta_5^{A^{tco}} - \beta_6^{A^{tco}} - 1 = 1$. $\mu_4^{A^{tco}} = 1$, $\mu_3^{A^{tco}} = 2$, $\mu_2^{A^{tco}} = 6$, $\mu_1^{A^{tco}} = 6$, $\mu^{A^{tco}} = (0, 1, 1, 2, 6, 6)$.

V. Conclusion

In this paper, a method is proposed to create a diagram that represents the outer chain movement of beta number in the active James diagram. A special case when the length = [1, 0, 0,…] for any e was presented to illustrate the rule for designing a new diagram, a rule to find the new diagram of b_1, b_2, b_3,… b_{e-1}, was revealed. An introduction point found after divided Active partition into two parts.

Table 2
Intersection point when μ = (93, 73, 33, 1) and e=3 in (a) James abacus (b) Diagram Atco.

A.D	-	o	-	-	o	o	o	-	-	-	-	o	o	o	-	-	o	o	o	-	-						
b_1	o	-	o	-	-	o	o	o	-	-	-	-	o	o	o	-	-	o	o	o	-	-	-	-			
b_2	o	o	-	o	-	-	o	o	o	-	-	-	-	o	o	o	-	-	o	o	o	-	-	-	-	-	-

(a)

.D	o	-	o	-	o	-	-	-	o	o	-	-	-	o	o	o	o	-	-	o	-						
b_1	-	o	o	o	-	-	-	o	-	o	-	o	-	o	o	o	-	-	-	o	-	o	-	-			
b_2	o	-	-	o	-	o	o	o	-	o	-	o	-	o	-	-	-	o	o	o	-	o	-	-	-	-	-

(b)

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