Stability of Linear Delay Differential Equations using Modified Algebraic Approach

K.H. Mohammedali[1], N.A. Ahmad[2]
[1]School of Mathematical Sciences, Universiti Sains Malaysia, 11800, Penang, Malaysia.
[2]School of Mathematical Sciences, Universiti Sains Malaysia, 11800, Minden, Penang, Malaysia.
khm13_mah009@student.usm.my

***Abstract*—An algebraic approach have been developed to study the stability of delay differential equations with m-retarded arguments, each of them is a multiple of fixed unknown time lag. The method has its basis upon transforming the characteristic equation related to the delay differential equation into an equivalent system of two algebraic equations in order to evaluate the value of the time lag which ensures the stability of the delay differential equation.**

***Index Terms*—Stability of Delay Differential Equations; Characteristic Equation; Linear Delay Differential Equations.**

I. INTRODUCTION

Delay differential equations (DDE's) are popular tools used by scientists in modeling real life systems. The forms of DDE models are usually proposed by investigators based on their interpretation to the system under consideration. In most cases, the parameters of the DDE models are unknown in which these parameters are often have important scientific interpretations and hence it is necessary to infer their values. Also it is a good calibration of the models formed as DDE if the solutions of the DDE fit the dynamical systems phenomena with certain parameter values, but in most cases this solution is so difficult to be evaluated and therefore it may be sufficient to study the stability of such solutions for increasing time without evaluating this solution explicitly, i.e., study its behavior for increasing time.

Among the most popular methods that may be used to study the stability of linear and nonlinear ordinary differential equations, in general and DDE's, in particular, are those which are based on analyzing certain functions called the Lyapunov function or by analyzing the characteristic equation related to the differential equation, in which so many analytical methods and numerical algorithms are proposed in literatures, [1].

Time delays are so many encountered in various real life systems, such as electric, pneumatic and hydraulic networks, chemical processes, etc. Therefore, the existence of time lags, regardless they are presented in the state or/and control may cause system's response to be undesirable, or even so system instability. Hence, as a consequence, the problem of stability analysis for such class of systems are one of the main interests for many researchers, since in general the time delay factors make the analysis much more complicated, [5], [15].

A large interest in those numerical and analytical studies of stability properties of linear DDE's have been paid recently, [13], [17]. Stability analysis of DDE's is in particular relevant in control theory, where one cause of the delay is the finite speed. Also, good study and characterization of DDE's and its stability based on its characteristic equations may be found in [15] and more mathematical demand treatments are given in [4].

Hsu C. S. in 1970 [10] study the stability of retarded DDE's with one delay which is based on finding a particular values of the time delay to ensure the existence of a pure imaginary roots of the characteristic equation.

Jury E. I. and Zeheb E. in 1983 [12] proposed an algebraic method to obtain the relevant values of certain control gain of a multivariable feedback system to be stable, in which the basis of this method involves an algebraic solution of two equations obtained from the real and imaginary parts of the characteristic equation of the system.

A good exposition of delay equations as well as the study of their stability properties based on their characteristic equations can be found in [17]. A mathematically more demanding treatment is found in [8]. Engelborghs et al. in 2002 [7] studied a multistep time-integration based numerical method to compute the characteristic roots .Hassard in 1997 [9] gave a formula that counts the number of unstable roots of the characteristic equation of DDE's. The method presented in [14] and [16] is also based on root counting and clustering to find stability regions by defining the kernel and offspring curves and observing an important invariance property of root crossings, they can completely characterize the stability regions. Asl and Ulsoyin 2003 [2] presented a new analytic approach to obtain the complete solution for systems of delay equations based on a Lambert-function expansion. Breda et al. in 2004 [3] propose a technique to compute the rightmost characteristic roots in which their method is based on the discretization of the infinitesimal generator of the solution operator semigroup and the approximation of the roots is obtained by a large sparse standard eigenvalue problem. Insperger and Stépánin 2002 [11] extended the method of semidiscretization to the study of DDE's. Semidiscretization utilizes the exact solution of linear systems over a short time interval to construct the mapping of a finite dimensional state vector for the system with time delay. Chen et al. in 1997 [6] reformulate the characteristic equation as an equation in a single unknown and use a robust numerical technique to solve this equation. Butcher et al. in 2004 [4] study the stability properties of delay-differential equations with time-periodic parameters by employing a shifted Chebyshev polynomial approximation in each time interval with length equal to the delay and parametric excitation period, the system is reduced to a set of linear difference equations for the Chebyshev expansion coefficients of the state vector in the previous and current intervals. This defines a linear map which can be thought of as the "infinite dimensional Floquet transition matrix".

In this paper, the method of Jury E. I. and Zeheb E. in composition with tau-decomposition method will be modified and used to study the stability of delay differential equations by evaluating the value of the time lag which ensures the stability of the differential equation.

II. Preliminaries and Basic Concepts

We start this section first with the formulation and the general outlines of the algorithm for determining the desired gain of multivariable feedback systems presented by Jurey E. I. and Zeheb E. [12]. Let G(s) denote the transfer function matrix of a linear time-invariant system with n-inputs and n-outputs. The feedback system is formed by connecting each output to an input and the difference between the two signals is fed through a control gain, K, which is common to all loops. The characteristic equation may be formulated according to the following:

$$P(s,K) = det[gI_n - G(s)] = 0 \quad (1)$$

where In is the nth order identity matrix and g =-1/K. Clear that, (1) defines an algebraic function s(K) which is assumed that without loose of generality it is irreducible over the field of rational functions. The problem now is reduced to obtain all feasible intervals of K, such that the values of s(K) are confined to the open left half complex s-plane (which is the stability condition)

A. *Algorithm (1), [12]*

i. From Equation (1) with pure imaginary value s = iw, we get two algebraic equations with two variables K and w, namely:

$$Re[P(iw,K)] = 0 \quad (2)$$

$$Im[P(iw,K)] = 0 \quad (3)$$

ii. Sensor assembly

iii. Solve Equation (2) and (3) for K., in which this step may be carried out systematically or numerically, if desired, and denote the real finite solutions (where it is assumed that both K and w are real) by K1, K2, …, Km with K1< K2< … < Km

iv. The values K_1, K_2, …, K_m will divides the real axis of K into m + 1 intervals. Then choose from each interval an arbitrary value and denote these arbitrary values by $K^{(0)}$, $K^{(1)}$, …, $K^{(m)}$.

v. Check the zeros of $P(s,K^{(j)})$, j = 0,1,…,m; for stability condition, where the complete open interval in which $K^{(j)}$ is located is a feasible interval rendering a stable system if and only if all the zeros of $P(s,K^{(j)})$, j = 0, 1, …, m; lie in the open left half of the complex s-plane, which is ascertained using the Routh-Hurwitz criterion.

The validity of the above algorithm is ensured also from [12], since it is evident that K_1, K_2, …, K_m for which eqs. (2) and (3) are satisfied for some real w are value of K for which at least one branch of the root locus of the algebraic function s(K) intersects or touches the imaginary axis of the s-plane. Therefore, since the zeros of the polynomial are continuous functions of its coefficients so that each branch of the root locus is a continuous curve, a zero s0(K) of f(s,K) can possibly move from the right half to the left half of the s-plane, or vice versa, only for the values K from the set {K_1, K_2, …, K_m}. Hence, the complete interval between successive values of Kj, j = 1, 2, …, m; renders either a stable or unstable system. It follows that f(s,K), $K_j < K < K_{j+1}$ is either a strict Hurwitz polynomial or not. Therefore, it suffices to check one arbitrary point in each of the intervals defined by the set {K_1, K_2, …, K_m} to determine all the real intervals of K for which the feedback system is stable.

Remark (1), [12], [15]:

1. The stable intervals of K are open intervals and the values of the end points of these intervals are K_1, K_2, …, K_m render unstable feedback system.
2. The set {K_1, K_2, …, K_m} and hence the set of check points {$K^{(0)}$,$K^{(1)}$,…,$K^{(m)}$} can be reduced in some cases by noting the root locus of the pertinent branch may only touch the imaginary axis of the s-plane, and does not intersect it. The related mathematical condition of this case is given by:

   $$Re\left[\frac{\partial P/\partial K}{\partial P/\partial s}\right]_{\substack{s=jw_s \\ K\in\mathbb{R}^+}} = 0 \text{ (of odd multiplicity)}$$

   where:

   $$dP = \frac{\partial P}{\partial s}ds + \frac{\partial P}{\partial K}dK\bigg|_{\substack{s=jw_s \\ K\in\mathbb{R}^+}} = 0$$

   and it is noted that if $\frac{\partial P}{\partial K} = 0$ is of even multiplicity, then this pertains to zero crossing of the imaginary axis. While when $\frac{\partial P}{\partial K} = 0$ is of odd multiplicity, then the real values of K_0 and w_0 are such that they do not define intervals which have to be checked for stability (or instability), because there is no crossing of a zero from right to left at the jw-axis or vice versa, hence we can ignore these K values. Furthermore, the two adjacent intervals are either both stable (except the point $K = K_0$) or both unstable, and it is suffices to choose a single check point different than K_0 determined from the next real values of K for both intervals.
3. The algorithm is also applicable, with no additional complexity, to similar problems with constraints on the allowable values of the gain K, or to somewhat more general problems which was not considered explicitly by [15].

The necessity and sufficient condition for stability for all values of gain for the multivariable feedback system may be stated next.

Let $P(s,K)$ of (1) be written as follows:

$$P(s,K) = a_0(K)s^j + a_1(K)s^{j-1} + \ldots + a_m(K) \tag{4}$$

For simplicity assume the following [12]:

1. Equation (4) is obtained after multiplying by the common divisor so that $a_i(K)$, i= 0,1,…,m; are polynomials in K.
2. The form $P(s,K)$ is irreducible.
3. $a_0(K) \not\equiv 0$, $a_m(K) \not\equiv 0$; otherwise we always treat $P(s,K)$ of reduced order.
4. m/2 is even (where the derivation of the case m/2 is odd, (m−1)/2 is even and (m−1)/2 is odd will be evident and need not be repeated.

Now, using (2) and (3) and noting (4), we have:

$$P_1 = a_0(K)w^j - a_2(K)w^{j-2} + \ldots + a_j(K) = 0 \tag{5}$$

$$P_2 = [a_1(K)w^j - a_3(K)w^{j-4} + \ldots + a_{j-1}(K)w = 0 \tag{6}$$

B. Theorem (1), [12]

For a multivariable feedback system with characteristic Equation (4), which is open-loop stable, to be stable for all values of the gain, it is necessary and sufficient that a real solution to (5) and (6) does not exist.

Now, in order to modify the above approach for systems of delay differential equations, consider:

$$f\left(t, x(t), \ldots x\left(g(t)\dot{x}(t), \dot{x}(g_1)\right), \ldots \dot{x}(g_m)\right) = 0 \tag{7}$$

with initial condition:

$$x(t) = \varphi(t),\ \ t_0 - r < t < 0, \varphi \in E_D$$

where $t_0 \in \mathbb{R}^+$, $m \in \mathbb{N}$, $g_i(t,r)$, i = 1,2,…,m; are given functions representing the retarded arguments and it will be assumed that $t_0 - r \le g_i(t,r) \le t_0$, j = 1,2,…,m; t > 0 and for some constant r > 0, which is called the time lag; x and $f: (0,\beta) \times E_D \to R^n$ are n-vector valued functions, $\beta \in \mathbb{R}^+$. Assume that *f* satisfies the conditions of the existence and uniqueness of solutions.

For the rest of this paper, it is necessary to recall the following basic definitions in stability theory:

Definition (1), [1]:

The trivial solution of (7) is stable at $t_0 > \alpha$ in the sense of Lyapunov if for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$, such that whenever $\|\varphi\| < \delta$ it follows that the solution $x(t, t_0, \varphi)$ exist on $[t_0 - r, \infty)$ and $\|x(t,t_0,\varphi)\| < \varepsilon$, for all $t_0 - r \le t$, where $\| \, . \, \|$ is the supremum norm.

Otherwise, the trivial solution is said to be unstable at t_0.

Definition (2), [1], [18]:

The trivial solution of (7) is said to be asymptotically stable at $t0 > \alpha$ if it is stable and there exists $\delta_1 = \delta_1(t) > 0$, such that whenever $\|\varphi\| < \delta_1$, then $\lim_{t\to\infty} x(t, t_0, \varphi) = 0$.

III. Stability of Linear Ordinary delay Differential Equations

One of the special forms of (7) is the system of linear ordinary differential equations with deviating arguments, retarded and neutral, is defined by:

$$\dot{x}(t) = \sum_{j=1}^{m} A_j x(t - r_j) + \sum_{j=1}^{m} B_j \dot{x}(t - r_j) \tag{8}$$

where $r_j \ge 0$, for all j=1,2,…,m; and $x(t) \in R^n$. Let $\overline{x}(t)$ be any solution of (8) and upon substituting the transformation $z(t) = x(t) - \overline{x}(t)$ will yields to:

$$\dot{z}(t) = \sum_{j=1}^{m} A_j z(t - r_j) + \sum_{j=1}^{m} B_j \dot{z}(t - r_j) \tag{9}$$

Therefore if z(t) = 0 is stable or unstable or asymptotically stable solution of (9), then the same is true for every solution of (8) and hence it is suffices to study he stability of the trivial solution of (8).

Assume a solution of (8) is of the form $x(t) = Ve^{\lambda t}$, where V is an n-vector and λ has a complex constant value. Hence, by substituting $x(t) = Ve^{\lambda t}$ in (8), the following algebraic equation is obtained:

$$\left[\lambda I - \sum_{j=1}^{m} A_j e^{-r_j \lambda} - \sum_{j=1}^{m} B_j \lambda e^{-r_j \lambda}\right] e^{\lambda t} V = 0$$

and in order to have a nontrivial solution, one must have:

$$\det\left[\lambda I - \sum_{j=1}^{m} A_j e^{-r_j \lambda} - \sum_{j=1}^{m} B_j \lambda e^{-r_j \lambda}\right] = 0 \tag{10}$$

This equation is called the characteristic equation, which is a polynomial of degree m that may be denoted by $P(\lambda, e^{-r_j \lambda})$. If the roots of (10) could be found, then the stability of the system linear ordinary DDE (8) may be determined based on the following theorems:

Theorem (2), [18]:

If all the roots of the characteristic equation (10) have negative real parts, then the trivial solution of (8) is asymptotically stable.

Theorem (3), [18]:

If at least one root has a positive real part, then the trivial solution of (8) is unstable.

Theorem (4), [18]:

If there are simple purely imaginary roots and the remaining roots have negative real parts, then the trivial solution of (8) is stable.

The above theorem ensures that the negativeness of the real parts plays an important role depending on the asymptotic behavior theorem of the stability of the ordinary differential equations.

IV. THE MODIFIED ALGEBRAIC APPROACH

This approach consists of the evaluation of the solution of two algebraic equations resulting from the real and imaginary parts of the complex characteristic equation or polynomial related to the delay differential equation. This approach is based on that approach proposed by Jury E. I. and Zeheb E. [12], which is originally established for determining the gain of multivariable feedback control systems that will be modified and improved here to study the stability of the system of linear DDE's by evaluating the value of the time lags which ensure the stability of the system under consideration.

Now, consider first in particular the system of retarded delay differential equations:

$$\dot{x}(t) = f(x(t), x(t-r), ..., x(t-mr)) \tag{11}$$

where r, t > 0 and $m \in \mathbb{N}$. Suppose that the characteristic equation related to eq.(11) is given by:

$$P(\lambda, r) = 0 \tag{12}$$

Therefore, in order to use the utilities of the complex numbers, letting $x(t) = Ve^{\lambda t}$, with the assumption that λ is pure imaginary, i.e., λ = iw, $w \in R$. Then the characteristic equation (12) may be rewritten as in the following complex function:

$$P(iw, r) = \mathrm{Re}[P(w, r)] + i\,\mathrm{Im}[P(w, r)] = 0$$

which implies from the properties of the complex numbers that:

$$\mathrm{Re}[P(w, r)] = 0 \tag{13}$$

$$\mathrm{Im}[P(w, r)] = 0 \tag{14}$$

Equations (13) and (14) represent an algebraic system of two equations with two variables w and r which may be solved to find them to ensure the stability of the system of DDE's given by (11).

In addition, since the procedure of finding the characteristic roots of (12) or equivalently of (13) and (14) is not simple, because (12) is an exponential polynomial. Thus, to check the direction of the root loci to determine whether the roots cross from the right half plane to the left half plane, or vice versa.

The characteristic equation in case of delay system contains an exponential terms of $-\lambda r$, which renders the system of algebraic Equations (13) and (14) which are also so difficult to be solved. Hence, in order to find r, we introduce a new variable T by replacing $e^{-\lambda r}$ in Equation (12) by $\frac{(1-\lambda T)^2}{(1+\lambda T)^2}$ and therefore will be not need to use Routh-Hurwitz criteria to study the stability of the system.

Now, rewrite the characteristic Equation (12) as follows:

$$P(\lambda, T) = a_0(T)\lambda^j + a_1(T)\lambda^{j-1} + ... + a_j(T) \tag{15}$$

where $a_0, a_1, \dots, a_j$ are polynomials in T and it is assumed that P is irreducible, i.e., $a_0(T) \not\equiv 0$, $a_j(T) \not\equiv 0$. Therefore, with λ=iw, the real and imaginary parts of (15) when j is even are given by:

$$P_1(w, T) = a_0(T)w^j - a_2(T)w^{j-2} + a_j(T) = 0 \tag{16}$$

$$P_2(w, T) = [a_1(T)w^{j-2} - a_3(T)w^{j-4} + ... + a_{j-1}(T)w = 0 \tag{17}$$

and vice versa if j is odd.

The next theorem give the necessary condition for stable solutions of linear DDE's presented by (11), which have been proved similarly to the proof of Theorem (1) given in [12] but with some modification.

Theorem (5):

If P_1 and P_2 have no real solutions, then (11) with the characteristic equation (15) is stable for all real values of the delay arguments.

Proof:

Since P_1 and P_2 are the real and imaginary parts of f with λ=iw, with real finite solutions T and w and suppose the solution for T to be the set$\{T_1, T_2, \dots, T_m\}$; with $T_1 < T_2 < \dots < T_m$ and the system (10)-(11) is satisfied for real w are those values of T for which at least one branch of the root locus of the resulting algebraic polynomial in T intersects (or touches) the imaginary axis of the w-plane.

Since the zeros of a polynomial are continuous functions of its coefficients, so that each branch of the root locus is a continuous curve and a zero can possibly move from the right half of the w-plane to the left half of the w-plane, or vice versa, only for values of T from the set $\{T_1, T_2, \dots, T_m\}$.

Hence, the complete intervals between the successive values T_i, i= 1,2,…,m; reaches either a stable or unstable system for certain values of r.

Thus no real solution to the system (16) and (17) exists, which means that none of the branches of the algebraic equations of T for which P(w,T) = 0 cross or touches the imaginary axis of the w-plane.

Therefore, if the system is stable for one value of T, i.e., its eigenvalues are clustered in the open left half of the w-plane and it will remains stable for all values of T.

Remark (2):

The new approach may be used to analyze the stability of the retarded linear delay systems and then improved for neutral systems with single or multi delays. The procedure of this approach may be stated and summarized as follows:

1. Substitute $\lambda = iw$ in Equation (6) and then evaluate the algebraic system (13)-(14) with $e^{-\lambda r} = \frac{(1-\lambda T)^2}{(1+\lambda T)^2}$ that is equivalently reduced to:

$$\left.\begin{array}{l}Re[P(w,T)] = 0\\ Im[P(w,T)] = 0\end{array}\right\} \quad (18)$$

2. Solve the resulting system (18), if possible, for non-negative real values w. The solutions may be denoted by the ascending set $\{T_1, T_2, \ldots, T_m\}$.
3. From the properties of the field of real numbers, the values T_1, T_2, …, T_m divide the real line T into m + 1 intervals (T_i, T_{i+1}), i= 1, 2, …, m.
4. Choose an arbitrary value in each interval, where these values may be denoted by $T^{(0)}$, $T^{(1)}$, …, $T^{(m)}$.
5. Check the zeros of $P(w,T^{(i)})$, i=0,1,…,m for stability.

To illustrate the present approach of this paper, we will consider first in example (1) retarded DDE which is considered and studied by [10], in which the author has shown that the interval of the time lag to ensure the asymptotic stability of the system is $(0, \frac{2\pi}{3\sqrt{3}})$.

Example (1):

Consider the retarded argument DDE with r, where r > 0:

$$\dot{x}(t) + x(t) + 2x(t-r) = 0, \; t > 0 \quad (19)$$

and to find the values of r for which (19) is stable.

First of all, the characteristic equation is obtained by assuming that $x(t) = Ve^{\lambda t}$, $\lambda \in \mathbb{C}$. Hence $\dot{x}(t) = \lambda V e^{\lambda t}$, $x(t-r) = Ve^{\lambda(t-r)}$ and thus we obtain:

$$P(\lambda, r) = \lambda + 1 + 2e^{-\lambda r} = 0 \quad (20)$$

Now, letting λ=iw (pure imaginary zeros), implies to:

$$iw + 1 + 2e^{-iwr} = 0$$

and recalling that $e^{i\theta} = \cos\theta + i\sin\theta$, which implies that (20) may be rewritten as:

$$iw + 1 + 2(\cos(wr) - i\sin(wr)) = 0$$

Hence:

$$[1 + 2\cos(wr)] + i[w - \sin(wr)] = 0$$

which is an algebraic equation. Hence the equivalent algebraic systems in w and r are given by:

$$1 + \cos(wr) = 0$$

$$w - 2\sin(wr) = 0$$

This non-linear system may be solved to find w and r, or by letting $e^{-\lambda r} = \frac{(1-\lambda T)^2}{(1+\lambda T)^2}$ in the characteristic Equation (20) yields to:

$$\lambda + 1 + 2\frac{(1-\lambda T)^2}{(1+\lambda T)^2} = 0$$

or:

$$\begin{aligned}P(\lambda, T) &= \lambda(1+\lambda T)^2 + (1+\lambda T)^2 + 2(1-\lambda T)^2\\ &= \lambda^3 T^2 + (2T + 3T^2)\lambda^2 + (1-2T)\lambda + 3 = 0\end{aligned} \quad (21)$$

It is remarkable that studying the stability of (21) using Routh-Hurwitz method is so difficult since it is a polynomial of the fifth degree in w with unknown coefficients.

Now, setting in (21), λ = iw, will overpass this problem and give:

$$(iw)^3 T^2 + (2T + 3T^2)(iw)^2 + (1-2T)iw + 3 = 0$$

and hence:

$$[3 - (2T + 3T^2)w^2] + i[(1-2T)w - w^3 T^2] = 0$$

Therefore the following system of algebraic equations will be obtained:

$$3 - (2T + 3T^2)w^2 = 0 \quad (22)$$

$$(1-2T)w - w^3 T^2 = 0 \quad (23)$$

Now, solving (22) we get:

$$w = \mp\sqrt{\frac{3}{2T + 3T^2}}$$

By substituting the value of w in (23), and solving the resulting equation for T, we get $T_1 = 1/3$ and $T_2 = -1/3$, in which the only nonnegative value of T will be used which is $T_1 = 1/3$; and hence implies that w_1= 1.7321.

Now, the value of T_1 will divide the nonnegative real axis into two intervals $(0,T_1)$ and (T_1,∞) and therefore we may choose an arbitrary values over each interval, say $T_1^{(0)} = 0.1$ and $T_1^{(1)} = 1$.

Finally, substituting $T_1^{(0)}$ in the characteristic equation (21) will give:

$$P(\lambda, T_1^{(0)}) = 0.01\lambda^3 + 0.23\lambda^2 + 0.8\lambda + 3 = 0$$

which has the roots:

$$\lambda = \begin{pmatrix} -19.71387301 \\ -1.64306349+3.53808595i \\ -1.64306349-3.53808595i \end{pmatrix}$$

which is clear that all the roots of λ have negative real parts, while substituting $T_1^{(1)}$ in (21) yields to:

$$P(\lambda, T_1^{(1)}) = \lambda^3 + 5\lambda^2 - \lambda + 3 = 0$$

which has the roots:

$$\lambda = \begin{pmatrix} -5.294 \\ 0.148+0.738i \\ 0.148-0.738i \end{pmatrix}$$

and it is clear that two roots have positive real parts.

As a result, the corresponding value of r with T ∈ (0,1/3) which stabilizes the retarded DDE (21) may be evaluated now which is found to be equal to r = 1.2092 and therefore the time lag interval of asymptotic stability is (0,1.2092), which is agree with the result given in [18].

Example (2):

Consider the retarded delay differential equation with two delays r and 2r, where r > 0:

$$\dot{x}(t) + x(t-r) + x(t-2r) = 0\ ,\ t > 0 \qquad (24)$$

and to find the values of r for which (24) is stable.

First of all, the characteristic equation is obtained by assuming that $x(t) = Ve^{\lambda t}$, $\lambda \in \mathbb{C}$. Hence $\dot{x}(t) = \lambda V e^{\lambda t}$, $x(t-r) = Ve^{\lambda(t-r)}$, $x(t-2r) = Ve^{\lambda(t-2r)}$ and thus we get:

$$P(\lambda, r) = \lambda + e^{-\lambda r} + e^{-2\lambda r} = 0 \qquad (25)$$

Now, let $\lambda = iw$ (pure imaginary zeros), then:

$$iw + e^{-iwr} + e^{-2iwr} = 0$$

and recalling that $e^{i\theta} = \cos\theta + i\sin\theta$, which implies that (25) may be rewritten as:

$$iw + \cos(wr) - i\sin(wr) + \cos(2wr) - i\sin(2wr) = 0$$

Hence:

$$[\cos(wr) + \cos(2wr)] + i[w - \sin(wr) - \sin(2wr)] = 0$$

which is an algebraic equation. Hence the equivalent algebraic systems in w and r is given by:

$$\cos(wr) + \cos(2wr) = 0$$

$$w - \sin(wr) - \sin(2wr) = 0$$

and it is clear that it is very difficult to be solved analytically or may be solved using computer programs, but it will have many roots. Therefore, letting $e^{-\lambda r} = \frac{(1-\lambda T)^2}{(1+\lambda T)^2}$ in the characteristic (14) yields to:

$$\lambda + \frac{(1-\lambda T)^2}{(1+\lambda T)^2} + \frac{(1-\lambda T)^4}{(1+\lambda T)^4} = 0$$

or:

$$P(\lambda, T) = \lambda(1+\lambda T)^4 + (1-\lambda T)^2(1+\lambda T)^2 + (1-\lambda T)^4 \qquad (26)$$
$$= T^4\lambda^5 + (4T^3 + 2T^4)\lambda^4 + (6T^2 - 4T^3)\lambda^3 + (4T + 4T^2)\lambda^2 + (1-4T)\lambda + 2 = 0$$

It is remarkable that studying the stability of (26) using Routh-Hurwitz method is very difficult since it is a polynomial of the fifth degree in w with unknown coefficients.

Now, setting in (26), $\lambda = iw$, will overpass this problem and give:

$$T^4(iw)^5 + (4T^2 + 2T^4)(iw)^4 + (6T^2 - 4T^3)(iw)^3 + (4T+4T^2)(iw)^2 + (1-4T)(iw) + 2 = 0$$

and hence:

$$[(4T^3 + 2T^4)w^4 - (4T + 4T^2)w^2 + 2] + i[T^4w^5 - (6T^2 - 4T^3)w^3 + (1-4T)w] = 0$$

Therefore, the following algebraic system will be obtained:

$$(4T^3 + 2T^4)w^4 - (4T + 4T^2)w^2 + 2 = 0 \qquad (27)$$

$$T^4w^5 - (6T^2 - 4T^3)w^3 + (1-4T)w = 0 \qquad (28)$$

Now, solving (27) we obtain:

$$w = \begin{bmatrix} \mp 1/\sqrt{T(2+T)} \\ \mp 1/T \end{bmatrix}$$

By substituting $w = \mp 1/\sqrt{T(2+T)}$ in (28), and solving the resulting equation for T, we obtain T_1= 0.1547005, T_2=–2.155 and T_3= 0, in which the only nonnegative value of T and will be used which is T_1= 0.1547005; hence implies that w_1= 1.732.

Similarly, substituting in (28) $w = \mp 1/T$ will give T = 0, which is not feasible.

Now, the value of T_1 will divide the nonnegative real axis into two intervals $(0,T_1)$ and (T_1,∞) and therefore we may choose an arbitrary values over each interval, say $T_1^{(0)}$ = 0.1 and $T_1^{(1)}$ = 1.

Finally, substituting $T_1^{(0)}$ in the characteristic equation (26) will give:

$$P(\lambda, T_1^{(0)}) = 0.0001\lambda^5 + 0.0042\lambda^4 + 0.056\lambda^3 + 0.44\lambda^2 + 0.6\lambda + 2$$

which has the roots:

$$\lambda=\begin{pmatrix} -0.431101+2.274499i \\ -0.431101-2.274499i \\ -7.047599+9.398524i \\ -7.047599-9.398524i \\ -27.042600 \end{pmatrix}$$

which is clear that all the roots of λ have a negative real parts, while substituting $T_1^{(1)}$ in (26) yields to:

$$P(\lambda,T_1^{(1)}) = \lambda^5 + 6\lambda^4 + 2\lambda^3 + 8\lambda^2 - 3\lambda + 2$$

which has the roots:

$$\lambda=\begin{pmatrix} 0.233585+0.416732i \\ 0.233585-0.416732i \\ -0.280138+1.185358i \\ -0.280138-1.185358i \\ -5.906894 \end{pmatrix}$$

and it is clear that two roots have positive real parts.

As a result, the corresponding value of r which stabilizes the retarded DDE (24) equals to r = 0.6046 and therefore the interval of asymptotic stability is (0,0.6046).

V. Conclusion & Recommendations

Stability of differential equations may be affected by the existence of time lags. Therefore, the evaluation of the time lags seems to be necessary to ensure the stability of the DDE. Also, the values of r which stabilize the system depends on the real life system under consideration and therefore, the present approach seems to be reliable in comparison with the other methods used to evaluate the time lag.

Acknowledgment

This work is supported by the Fundamental Research Grant Scheme (FRGS) under the Ministry of Higher Education, Malaysia.

References

[1] Arivo O. et al. (eds.), “Delay Differential Equations and Applications”, Springer, Berline, 477-517, 2006.
[2] Asl F. M. and Ulsoy A. G., “Analysis of a System of Linear Delay Differential Equations”, Journal of Dynamic Systems, Measurement, and Control, Vol.125, 215-223, 2003.
[3] Breda D., Maset S. and Vermiglio R., “Computing the Characteristic Roots for Delay Differential Equations”, IMA Journal of Numerical Analysis, 24, 1-19, 2004.
[4] Butcher E. A., Ma H., Bueler E., Averina V. and SzabóZs., “Stability of Time-Periodic Delay Differential Equations Via Chebyshev Polynomials”, International Journal for Numerical Methods in Engineering, 59, 895-922, 2004.
[5] Campbell S. A., “Introduction to Delay Differential Equations”, Department of Applied Mathematics, University of Waterloo, 2007.
[6] Chen S. G., Ulsoy A. G. and Koren Y., “Computational Stability Analysis of Chatter in Turning”, Journal of Manufacturing Science and Engineering, 119, 457-460, 1997.
[7] Engelborghs K. and Roose D., “On Stability of LMS Methods and Characteristic Roots of Delay Differential Equations”, SIAM Journal of Numerical Analysis, 40, 629-650, 2002.
[8] Hale J. K. and Lunel S. M. V., Introduction to Functional Differential Equations, Applied Mathematical Sciences, 99, Springer Verlag, New York, 1993.
[9] Hassard B. D., “Counting Roots of the Characteristic Equation for Linear Delay-Differential Systems”, Journal of Differential Equations, Vol.136, 222-235, 1997.
[10] Hsu C. S., “Application of the Tau-Decomposition Method to Dynamical Systems Subjected to Retarded Follower Forces”, J. Applied Mechanics, 37, 259-266, 1970.
[11] Insperger T. and Stépán G., “Semidiscretization Method for General Delayed Systems”, International Journal for Numerical Methods in Engineering, 55, 503-518, 2002.
[12] Jury E. I. and Zeheb E., An Algebraic Algorithm for Determining the Desired Gain of Multivariable Feedback Systems”, Int. J. Control, Vol.37, No.5, 1081-1094, 1983.
[13] Nagy T. K., “Stability Analysis of Delay Differential Equations by the Method of Steps and Inverse Laplace Transform”, Differential Equations and Dynamical Systems, Vol.17, No. 1 & 2, 185-200, 2009.
[14] Olgac N., and Sipahi R., “An Exact Method for the Stability Analysis of Time-Delayed LTI Systems”, IEEE Trans. Autom. Control, 47(5), 793-797, 2002.
[15] Postlethwaite I. and Macfarlane A. G. J., A complex Variable Approach to the Analysis of Linear Multivariable Feedback Systems, Lecture Notes in Control and Information Science, 12, New York, Springer Verlag, 1979.
[16] Sipahi R. and Olgac N., “A Novel Stability Study on Multiple Time-Delay Systems (MTDS) Using the Root Clustering Paradigm”, Proceedings of the ACC, 2004.
[17] Stépán G., Retarded Dynamical Systems. Longman: Harlow, UK, 1989.
[18] Verriest E. I., New Qualitative Methods for Stability of Delay Systems, Kybernetika, Vol.37, No.3, 229-238, 2001